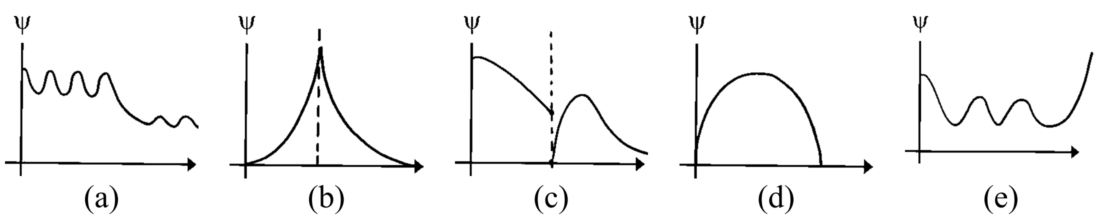
Problems

1. **(3 points) What is the *physical meaning* of the normalization condition?**
   Answer: Because the particle must be somewhere along the x-axis, the sum of the probabilities over all values of x must be 1: $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$. *Physically speaking, this is simply a statement that the particle exists all the time. Any wave function satisfying this equation is said to be normalized.*

2. **(4 points) Name two *observables* of a particle and state whether they are *sharp* or *fuzzy*.**
   Answer: An observable is any particle property that can be measured. For example, the position, momentum and total energy are all observables. Observables can be either fuzzy, where repeated measurements of the quantity yield different values (such as position or momentum), or sharp, where repeated measurements yield the same value (such as the total energy for stationary states).

3. **(3 points) Explain the difference between *phase* and *group* velocities for a wave packet.**
   Answer: A localized wave packet is made up of many frequency components. Typically, it's a sum of plane waves having a continuous distribution of frequencies. The speed at which a ***single*** *frequency component travels is called the phase velocity and is given by* $v_{phase} = \frac{\omega}{k}$. *The entire wave packet usually travels, as a whole, at a different velocity called the group velocity and given by* $v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0}$.

4. **(10 points; 2 each) Of the following functions, which are candidates for the Schrödinger wave function of an actual physical system? For those that are not, state why they fail to qualify.**



(a) Yes. It's ok, the requirement is that $\psi(x)$ *must be finite everywhere.*

(b) Yes. This wave function is still a candidate even though the slope is not continuous. The requirement is that $\frac{d\psi}{dx}$ *be continuous for a finite U(x). This choice of wave function just implies that the potential is not finite.*

(c) No. $\psi(x)$ *must be continuous everywhere.*

(d) Yes. $\psi(x)$ *can be zero outside of some range (think of the infinite square well).*

(e) No. $\psi(x)$ *must be finite everywhere but it's diverging for* $x \to \infty$.

5. **(5 points) Consider a particle in a box (a.k.a. the infinite square well). Calculate the probability that a particle, in the ground state, will be found in the middle half of the well (i.e. $[\frac{L}{4}, \frac{3L}{4}]$).**

The probability density is given by

$$P = \int_{L/4}^{3L/4} |\psi_1(x)|^2 dx = \left(\frac{2}{L}\right)\int_{L/4}^{3L/4} \sin^2\left(\frac{\pi x}{L}\right)dx = \left(\frac{1}{L}\right)\int_{L/4}^{3L/4} [1 - \cos\left(\frac{2\pi x}{L}\right)]dx$$

Where we've used the trig identity: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

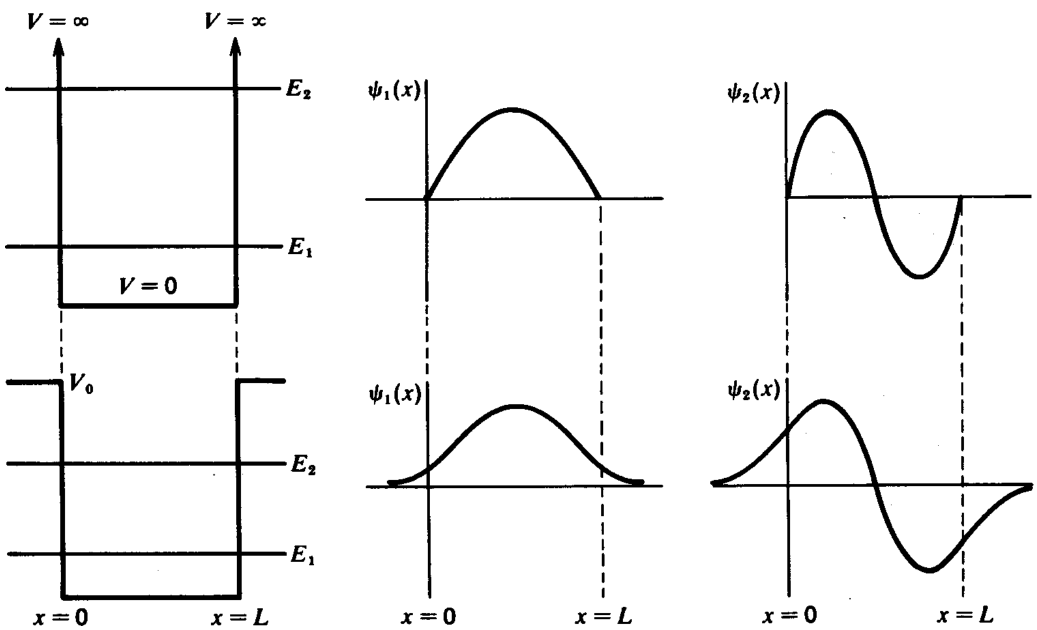
$$P = \left(\frac{1}{L}\right)\left[\frac{L}{2} - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\Bigg|_{L/4}^{3L/4}\right] = \frac{1}{2} + \left(\frac{1}{\pi}\right)$$

Numerically,

$$P = 0.818$$

Classically, we expect the particle to spend half its time inside half of the well (i.e. equal time in all parts of the well). For the ground state, we see that this is considerably larger than ½. If we were to repeat this calculation for the n^{th} state we would see that the result approaches the classical value ½ in the limit $n \to \infty$.

6. **(8 points; 2 each) Graphically compare the 1st and 2nd bound state of the infinite and finite square well by *carefully* sketching their wave functions in the space provided.**



7. **(5 points) Consider an electron bound in a harmonic oscillator, $E_{total} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, with an arbitrary uncertainty *a* ($\Delta x \sim a$). Using the uncertainty principle, (a) *estimate* the value of *a* that minimizes the total energy and (b) find this minimum energy and compare it to the exact harmonic oscillator ground state energy.**

Answer: *(a) Letting $\Delta x \sim a$, and using the uncertainty principle, we find that $\Delta p \sim \frac{\hbar}{2a}$. For this general estimation, let's assume that the uncertainty in momentum is on the same scale as the momentum itself, $\Delta p \sim p$. And so, we can write the total energy for the quantum harmonic oscillator in terms of 'a':*
$E_{total} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar^2}{8ma^2} + \frac{1}{2}m\omega^2 a^2$. *Since we want the value of 'a' that minimizes the energy, we set $\frac{dE}{da} = 0$ and solve for $a_{\min}$:*

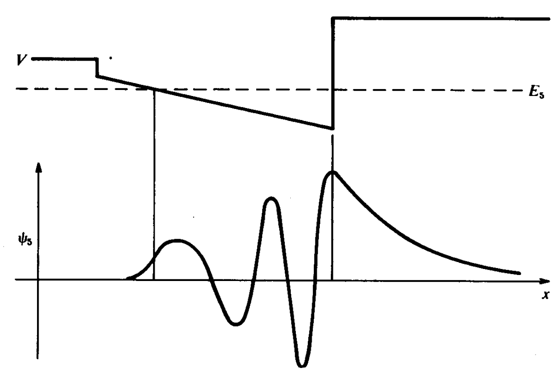
$$E_{total} = -\frac{\hbar^2}{4ma^3} + m\omega^2 a = 0 \quad \rightarrow \quad \frac{\hbar^2}{4ma_{\min}^3} = m\omega^2 a_{\min} \quad \rightarrow \quad a_{\min} = \sqrt{\frac{\hbar}{2m\omega}}$$

(b) Plugging back into the energy we find that

$$E_{\min} = \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right) + \frac{1}{2}m\omega^2\left(\frac{2m\omega}{\hbar}\right) = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega$$

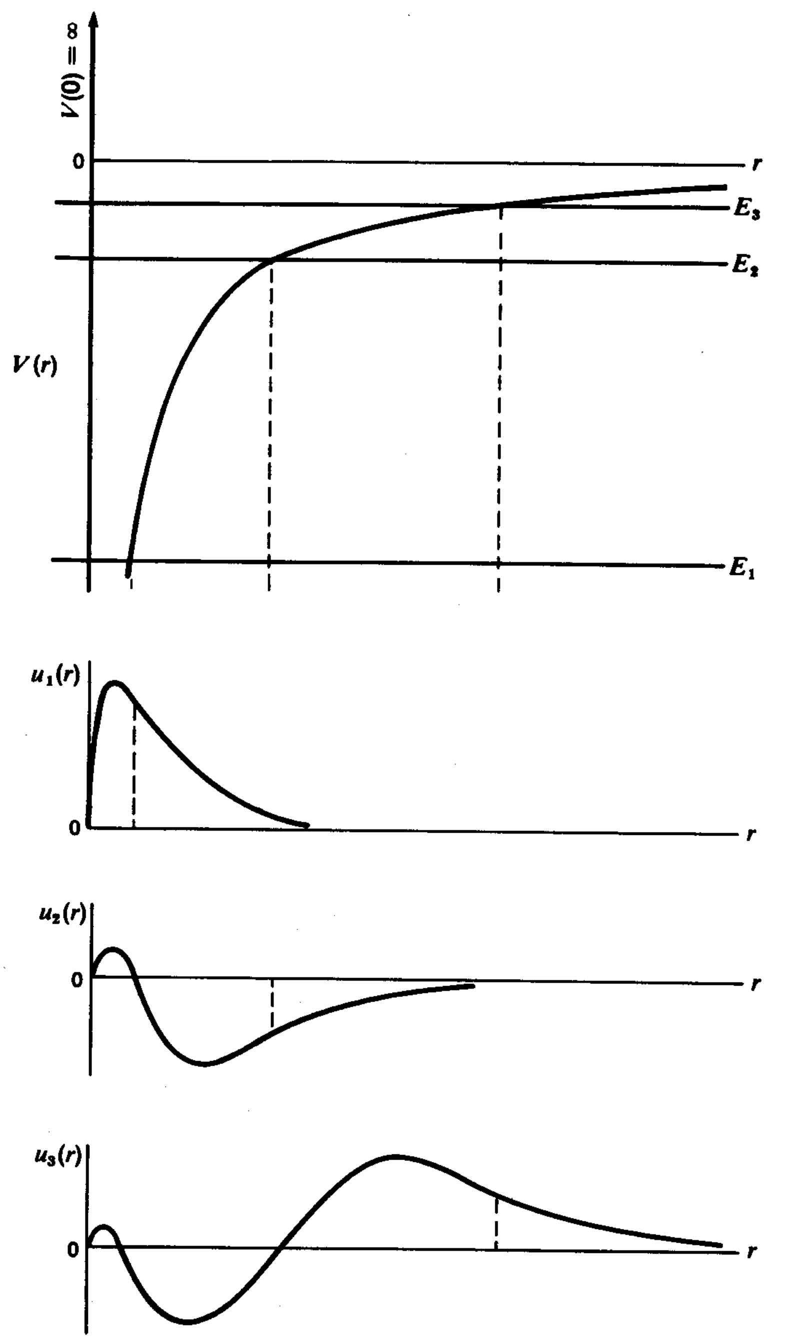
which is the value of the exact ground state energy.

8. **(3 points) The curve in the figure is alleged to be the plot of a computer-calculated wave function for the fifth energy level of a particle in the diagrammed one-dimensional potential well. Indicate the way in which the plot *fails* to be qualitatively correct.**



The asymmetry in the wave function is correct (the potential is not symmetric), the number of nodes is also correct (4 nodes for the 5th bound state), the relative wavelengths are also correct (shorter wavelength on the right side where the kinetic energy [E-U(x)] is bigger), BUT the relative amplitudes are NOT correct! We expect the amplitude to be inversely proportional to KE. So our wave function should have higher amplitude on the left side and lower on the right.

9. **(9 points; 3 each) Sketch careful, qualitatively accurate plots (on the space provided) for the 1st, 2nd, and 3rd bound state of a 1-D Coulomb potential. In your plots, don't forget to clearly mark the location of the classical turning points. Notice that $V(r \le 0) = \infty$. *Important: Check that your wave function has the correct symmetry, number of nodes, relative wavelengths, maximum values of amplitudes and relative rate of decrease outside the well.***

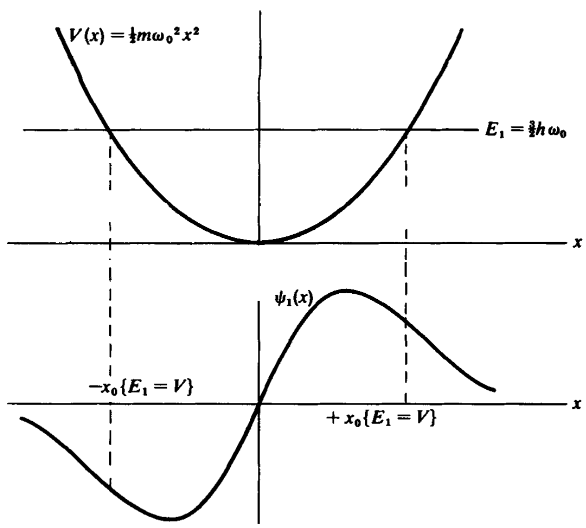


10. **(14 points) The 2nd bound state (n = 1) wave function of the harmonic oscillator is given by**

$$\psi_1(x) = \left(\frac{1}{2a\sqrt{\pi}}\right)^{1/2} 2\left(\frac{x}{a}\right) e^{-\frac{x^2}{2a^2}} \quad \textbf{where } a = \sqrt{\frac{\hbar}{m\omega}}.$$

(a) (2 points) *Carefully* sketch a qualitative but accurate plot of $\psi_1(x)$. Clearly mark the location of the classical turning points.
(b) (3 points) Show that the wave function is normalized.
(c) (6 points) Compute the quantum uncertainty in the position, Δx, in terms of *a*.
(d) (3 points) What is $< H >$?

Answer: (a) This is a plot of the potential with the 1st bound state shown. The classical turning points are defined as the places on the potential where the energy of the particle equals the potential energy.



(b) If the given wave function is normalized, then $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. *And so*

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \left(\frac{1}{2a\sqrt{\pi}}\right)\left(\frac{2}{a}\right)^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{a^2}} dx =$$

$$= \frac{4}{a^3\sqrt{\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{a^2}} dx = \frac{4}{a^3\sqrt{\pi}} \frac{a^2}{4}\sqrt{a^2\pi} = 1$$

Where we've made use of the integral: $\int_0^{\infty} u^2 e^{-\beta u^2} du = \frac{1}{4\beta}\sqrt{\frac{\pi}{\beta}}$.

(c) In order to find the quantum uncertainty in the position, Δx, we must compute $\Delta x = \sqrt{<x^2> - <x>^2}$, which is really two separate integrals.

Solving for $<x>$ first:

$$<x> = \int_{-\infty}^{\infty} \psi^*[x]\psi dx = \frac{4}{2a^3\sqrt{\pi}} \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{a^2}} dx = 0$$

Notice that the function x^3 is odd and the function $e^{-\frac{x^2}{a^2}}$ is even. Their product is therefore odd. We see that, just like in the homework problems, we are taking the integral of an odd function over all of space. In this case, the integral over the negative half-axis exactly cancels that over the positive half-axis.

For the calculation of $<x^2>$, however, the integrand is even and the two half-axes contribute equally, giving

$$<x^2> = \frac{4}{2a^3\sqrt{\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{a^2}} dx = \frac{4}{a^3\sqrt{\pi}} \int_{0}^{\infty} x^4 e^{-\frac{x^2}{a^2}} dx$$

$$<x^2> = \frac{4}{a^3\sqrt{\pi}} \frac{3a^4}{8} \sqrt{\pi a^2} = \frac{3}{2} a^2$$

Where we've made use of the integral: $\int_{0}^{\infty} u^4 e^{-\beta u^2} du = \frac{3}{8\beta^2}\sqrt{\frac{\pi}{\beta}}$.

And so

$$\Delta x = \sqrt{<x^2> - <x>^2} = \sqrt{\frac{3}{2}} a$$

(d) Asking for the expectation value of the Hamiltonian operator, $<H>$, is the same thing as asking for the energy of the state, which is simply $E_1 = \frac{3}{2}\hbar\omega$!

If for some reason, you didn't recognize this, you can still do the calculation fairly easily.

$$<H> = \int_{-\infty}^{\infty} \psi_1^*[H]\psi_1 dx = \int_{-\infty}^{\infty} \psi_1^* E_1 \psi_1 dx = E_1 \int_{-\infty}^{\infty} \psi_1 * \psi_1 dx = E_1 = \frac{3}{2}\hbar\omega$$

Here you needed to recognize that $[H]\psi$ is just a short hand notation of the Schrödinger equation (i.e. $[H]\psi_n = E_n \psi_n$).