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PHYSICS 410 — Prof. S. J. Gates

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This is a OPEN book examination. Read the entire examination before you begin to work. Be sure to read each problem carefully. Any questions should be directed to the proctor. There is an hour & fifty minute time limit. Show all of your work. Use the backs of pages if necessary or request an extra booklet. Be sure to complete the front page of the examination booklet including your name. Show all calculations needed to support your answers, where necessary. Most importantly, THINK before you start to calculate.

Problem (1.)

(a.) Using the inverse of the radial distance, (i.e. $r = u^{-1}$) and θ as the independent variable, leads to

$$K = \frac{1}{2} m \left\{ (\frac{d\,r}{d\theta})^2 \; + \; r^2 \right\} (\frac{d\,\theta}{dt})^2 \quad , \quad L \; = \; m\,r^2 \,\frac{d\,\theta}{dt} \quad ,$$

$$K = \frac{1}{2} m \left\{ \frac{1}{u^4}(\frac{d\,u}{d\theta})^2 \; + \; \frac{1}{u^2} \right\} (\frac{d\,\theta}{dt})^2 \quad , \quad L \; = \; m\,u^{-2} \,\frac{d\,\theta}{dt} \quad ,$$

and thus

$$\frac{K}{L^2} \; = \; \frac{1}{2\,m} \left\{ (\frac{d\,u}{d\theta})^2 \; + \; u^2 \right\}$$

This equation then implies

$$\frac{1}{2\,m} \left\{ (\frac{d\,u}{d\theta})^2 \; + \; u^2 \right\} \; = \; A_0 \, exp[\, 2\,\theta \,]$$

Since the right hand side of the equation involves an exponential, it is natural to make the ansatz

$$u(\theta) \; = \; \alpha_0 \; exp[\, \theta \,]$$

where α_0 is a constant. The equation will be solved it $\alpha_0 = \sqrt{m\,A_0}$

(b.) From the solution above

$$r(\theta) \;=\; \frac{1}{\sqrt{m\,A_0}}\; exp[\,-\,\theta\,]$$

If the angular momentum is given by $L_0 exp[-2(t/\tau_0)\,]$ then it must be the case that

$$\begin{aligned}
L_0\, exp[-2(t/\tau_0)\,] \;&=\; \left\{ \frac{1}{2\,A_0} \right\} exp[\,-\,2\,\theta\,]\,(\frac{d\,\theta}{dt}) \\
exp[-2(t/\tau_0)\,]d\,t \;&=\; \left\{ \frac{1}{2\,A_0\,L_0} \right\} exp[\,-\,2\,\theta\,]\,d\,\theta \\
-\,(\tau_0/2)d\,\{\,exp[-2(t/\tau_0)\,]\} \;&=\; -\;d\,\left\{ \frac{exp[\,-\,2\,\theta\,]}{4\,A_0\,L_0} \right\} \\
d\;\{\,exp[-2(t/\tau_0)\,]\} \;&=\; \left\{ \frac{1}{2\,\tau_0\,A_0\,L_0} \right\} d\;\{\,exp[\,-\,2\,\theta\,]\,\} \\
\int d\;\{\,exp[-2(t/\tau_0)\,]\} \;&=\; \left\{ \frac{1}{2\,\tau_0\,A_0\,L_0} \right\} \int d\;\{\,exp[\,-\,2\,\theta\,]\,\} \\
exp[-2(t/\tau_0)\,] \;-\; 1 \;&=\; \left\{ \frac{1}{2\,\tau_0\,A_0\,L_0} \right\} \{\,exp[\,-\,2\,\theta\,] \;-\; exp[\,-\,2\,\theta_0\,]\,\}
\end{aligned}$$

To make further progress, it is useful to choose θ_0 such that

$$exp[\,\theta\,] \;=\; \frac{1}{\sqrt{2\,\tau_0\,A_0\,L_0}}\, exp[\;(t/\tau_0)\,]$$

which leads to

$$\begin{aligned}
\theta(t) \;&=\; (t/\tau_0) \;-\; \tfrac{1}{2}\,\ln[2\,\tau_0\,A_0\,L_0\,] \\
r(t) \;&=\; \left[\sqrt{\frac{2\,\tau_0\,L_0}{m}} \right] exp[\,-\,(t/\tau_0)\,]
\end{aligned}$$

Now the if there is a potential $U(r,\,\theta)$ it must satisfy

$$-\,(\frac{\partial\,U}{\partial r}) \;=\; m\,\left[\frac{d^2\,r}{dt^2} \;-\; r\,(\frac{d\,\theta}{dt})^2 \right]$$

and when the expressions for $r(t)$ and $\theta(t)$ are used this implies

$$(\frac{\partial\,U}{\partial r}) \;=\; 0$$

This implies that $U(r,\, \theta) = U(\theta)$. Next there is the equation

$$
\begin{aligned}
-\left(\frac{\partial U}{\partial \theta}\right) &= m\, \frac{d}{dt}\left[r^2\, \frac{d\,\theta}{dt} \right] \\
&= \frac{m}{\tau_0}\, \frac{d}{dt}\left[r^2 \right] \;=\; \frac{2\,m}{\tau_0}\, r\, \left[\frac{d\,r}{dt} \right] \\
&= \frac{2\,m}{\tau_0}\left[\frac{1}{\sqrt{m\,A_0}}\; exp[\,-\,\theta\,] \right] \frac{d}{dt}\left[\frac{1}{\sqrt{m\,A_0}}\; exp[\,-\,\theta\,] \right] \\
&= \left[\frac{2}{\tau_0\,A_0} \right] exp[\,-\,\theta\,]\, \frac{d}{dt}\,[\; exp[\,-\,\theta\,]\,] \\
&= -\, \left[\frac{2}{(\tau_0)^2\,A_0} \right] exp[\,-\,2\,\theta\,]
\end{aligned}
$$

and this has the solution

$$
U(r,\, \theta) \;=\; -\, \left[\frac{1}{(\tau_0)^2\,A_0} \right] exp[\,-\,2\,\theta\,]
$$

Problem (2.)

(a.) To find the location of the center of mass, we first find the mass/length for each wire. The mass of each is M_w and the radius of each semi-circle is $r_0 = 4$. So that mass/length $= M_w/\pi r_0$. This means that we have using cylindrical coordinates

$$
\int d\,V \mu(\vec{r}) \;=\; \int d\rho\, \rho d\phi\, dz\;\, \mu(\vec{r}) \;=\; \int_0^\pi d\,\phi\; \frac{M_w}{\pi} \quad .
$$

where $\mu(\vec{r})$ is the mass per unit volume. So the center of mass for the wire in the x-y plane is given by

$$
\begin{aligned}
\vec{R}^{(1)}_{cm} &= \frac{1}{M_w} \int d\rho\, \rho d\phi\, dz\; \vec{r}\; \mu(\vec{r}) \\
\vec{R}^{(1)}_{cm} &= \frac{1}{M_w}\; \left[\frac{M_w}{\pi} \right] \int_0^\pi d\phi\; \vec{r} \;=\; \left[\frac{1}{\pi} \right] \int_0^\pi d\phi\; \vec{r}
\end{aligned}
$$

On the first piece of wire, we have

$$
\vec{r} \;=\; r_0\,[\,cos\phi\,\widehat{x} \;+\; sin\phi\,\widehat{y}\,]
$$

so that

$$
\begin{aligned}
\vec{R}^{(1)}_{cm} &= \left[\frac{r_0}{\pi} \right] \int_0^\pi d\phi\;\, [\,cos\phi\,\widehat{x} \;+\; sin\phi\,\widehat{y}\,] \\
&= \left[\frac{r_0}{\pi} \right] \widehat{y} \int_0^\pi d\phi\;\, [\,sin\phi\,] \;=\; \left[\frac{2}{\pi} \right] r_0\, \widehat{y}
\end{aligned}
$$

and thus for the second wire

$$\vec{R}^{(2)}_{cm} = \left[\frac{2}{\pi} \right] r_0 \, \widehat{z}$$

Finally to find the center of mass of the system

$$\begin{aligned}
\vec{R}^{(Tot)}_{cm} &= \left[\frac{M_w \, \vec{R}^{(1)}_{cm} \; + \; M_w \, \vec{R}^{(2)}_{cm}}{2M_w} \right] \; = \; \frac{1}{2} \left[\vec{R}^{(1)}_{cm} \; + \; \vec{R}^{(2)}_{cm} \right] \\
&= \left[\frac{1}{\pi} \right] r_0 \, [\, \widehat{y} \; + \; \widehat{z} \,]
\end{aligned}$$

(b.) To find the moment of inertia tensor for the system of wire we start with the definition of the moment of inertia tensor for the first wire

$$\begin{aligned}
\mathcal{I}^{(1)}_{i\,j} &= \int d\rho \, \rho d\phi \, dz \;\, \mu(\vec{r}) \, \left[|\vec{r}|^2 \delta_{i\,j} \; - \; r_i \, r_j \right] \\
&= \int_0^\pi d\,\phi \, \frac{M_w}{\pi} \, \left[|\vec{r}|^2 \delta_{i\,j} \; - \; r_i \, r_j \right] \\
&= \left[\frac{M_w}{\pi} \right] \int_0^\pi d\,\phi \, \begin{pmatrix} y^2 & -\, x\,y & 0 \\ -\, y\,x & x^2 & 0 \\ 0 & 0 & x^2 \, + \, y^2 \end{pmatrix} \\
&= \left[\frac{M_w \, r_0^2}{\pi} \right] \int_0^\pi d\,\phi \, \begin{pmatrix} sin^2\phi & -\, cos\phi \, sin\phi & 0 \\ -\, sin\phi \, cos\phi & cos^2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= M_w \, r_0^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .
\end{aligned}$$

This implies for the second wire

$$\mathcal{I}^{(2)}_{i\,j} \; = \; M_w \, r_0^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad .$$

and thus for the total moment

$$\mathcal{I}^{(Tot)}_{i\,j} \; = \; \mathcal{I}^{(1)}_{i\,j} \; + \; \mathcal{I}^{(2)}_{i\,j} \; = \; \frac{1}{2} M_w \, r_0^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad .$$

(c.) The rotational kinetic energy of the wire system is thus

$$T_{Rot} \; = \; \frac{1}{4} \, M_w \, r_0^2 \, \left[2(\omega_x)^2 \; + \; 3\,(\omega_y)^2 \; + \; 3\,(\omega_z)^2 \right]$$

Problem (3.)

Each captain states in their frame of reference the frequency of their running light is 430 trillion Hz. The data of the Vulcan scientist reads

Table 1: Sensor Data

	Mass	Frequency	Length
Enterprise	190 million kg	680 trillion Hz	1000 m
Warbird	200 million kg	720 trillion Hz	1250 m

(a.) If the scientist observed speed of the approach v_E of the Enterprise to Vulcan and the speed of the approach v_W of the Warbird to Vulcan, she could deduce the speed of approach of the Warbird observed from the deck of the Enterprise.

$$v_A = \left[\frac{v_E + v_W}{1 + \frac{v_E v_W}{c^2}} \right] \quad , \quad \beta_A = \frac{v_A}{c}$$

The formulae for the relativistic Doppler Effect is given by

$$f' = f \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}$$

If we define the ratio $f/f' = F$ this leads to

$$\beta = \left| \frac{1 - F^2}{1 + F^2} \right| = \left| \frac{(f')^2 - f^2}{(f')^2 + f^2} \right|$$

$$F_E = (68/43) \quad , \quad F_W = (72/43) \quad ,$$

$$v_E = c \left| \frac{(43)^2 - (68)^2}{(43)^2 + (68)^2} \right| \quad , \quad \beta_E = \frac{v_E}{c}$$

$$v_W = c \left| \frac{(43)^2 - (72)^2}{(43)^2 + (72)^2} \right| \quad , \quad \beta_W = \frac{v_W}{c}$$

(b.) The Vulcan scientist is not in the rest from of the Enterprise, so the mass she observes $(M_E)' = 190 \times 10^6$ kg is not the rest mass of the ship M_E^0. The relation between these is

$$M_E^0 = (M_E)' \sqrt{1 - (\beta_E)^2}$$

The mass $(M_E)''$ observed from the deck of the Warbird is related to the rest mass of the ship M_E^0 via

$$
\begin{aligned}
(M_E)'' &= \frac{M_E^0}{\sqrt{1 - (\beta_A)^2}} \\
&= (M_E)' \sqrt{\frac{1 - (\beta_E)^2}{1 - (\beta_A)^2}}
\end{aligned}
$$

Problem (4.)

(a.) To find the acceleration and velocity vectors of the airplane we see

$$
\begin{aligned}
\vec{V}_p &= - \rho_0 \, \omega_0 \, [\, sin(\omega_0 \, t)\widehat{x} \; + \; cos(\omega_0 \, t)\widehat{y}\,] \; - \; v_0 \, \widehat{z} \\
\vec{A}_p &= - \rho_0 \, (\omega_0)^2 \, [\, cos(\omega_0 \, t)\widehat{x} \; - \; sin(\omega_0 \, t)\widehat{y}\,]
\end{aligned}
$$

(b.) The airplane lands when $\hat{z} \cdot \vec{R}_p = 0$ and this occurs at the time $t = H_0/v_0$. If n denotes the number of complete rotations and f the fractional part then

$$
n \; + \; f \; = \; \frac{1}{2\pi} \left[\frac{\omega_0 \, H_0}{v_0} \right]
$$

(c.) The orthogonal unit vectors for your 'x-direction,' 'y-direction' and 'z-direction' may be denoted by $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$. For an observer on the ground these are written as

$$
\begin{aligned}
\hat{e}_1 &= - \left\{ \frac{\rho_0 \, \omega_0 \, [\, sin(\omega_0 \, t)\widehat{x} \; + \; cos(\omega_0 \, t)\widehat{y}\,] \, + \, v_0 \, \widehat{z}}{\sqrt{(\rho_0 \, \omega_0)^2 \; + \; (v_0)^2}} \right\} \\
\hat{e}_2 &= [\, cos(\omega_0 \, t)\widehat{x} \; - \; sin(\omega_0 \, t)\widehat{y}\,] \\
\hat{e}_3 &= \left\{ \frac{- \, v_0 \, [\, sin(\omega_0 \, t)\widehat{x} \; + \; cos(\omega_0 \, t)\widehat{y}\,] \, + \, \rho_0 \, \omega_0 \, \widehat{z}}{\sqrt{(\rho_0 \, \omega_0)^2 \; + \; (v_0)^2}} \right\}
\end{aligned}
$$

It is convenient to define φ_0 by

$$
tan\varphi_0 \; = \; \left\{ \frac{v_0}{\rho_0 \, \omega_0} \right\}
$$

so that

$$
\begin{aligned}
\hat{e}_1 &= - \, cos\varphi_0 \, \omega_0 \, [\, sin(\omega_0 \, t)\widehat{x} \; + \; cos(\omega_0 \, t)\widehat{y}\,] \, - \, sin\varphi_0 \, \widehat{z} \\
\hat{e}_2 &= [\, cos(\omega_0 \, t)\widehat{x} \; - \; sin(\omega_0 \, t)\widehat{y}\,]
\end{aligned}
$$

$$\hat{e}_3 = \; - \, sin\varphi_0 \, \omega_0 \, [\, sin(\omega_0 \, t)\widehat{x} \; + \; cos(\omega_0 \, t)\widehat{y} \,] \, + \, cos\varphi_0 \, \widehat{z}$$

(d.) To write Newton's Second Law in your frame of reference, it is important to note

$$\begin{aligned} \frac{d}{dt}\hat{e}_1 &= \; - \, \omega_0 \, cos\varphi_0 \, \hat{e}_2 \\ \frac{d}{dt}\hat{e}_2 &= \; \omega_0 \, [\, cos\varphi_0 \, \hat{e}_1 \; + \; sin\varphi_0 \, \hat{e}_3 \,] \\ \frac{d}{dt}\hat{e}_3 &= \; - \, \omega_0 \, sin\varphi_0 \, \hat{e}_2 \end{aligned}$$

The position vector for an object in your reference frame takes the form

$$\vec{\xi} \; = \; U \, \hat{e}_1 \; + \; V \, \hat{e}_2 \; + \; W \, \hat{e}_3$$

for some coordinates U, V, W. If the object has a mass of $\mathcal{M}$ you write

$$\begin{aligned} \vec{F} &= \; \mathcal{M} \, \frac{d^2}{dt^2}\vec{\xi} \; = \; \mathcal{M} \, \frac{d}{dt}\,\frac{d}{dt}\vec{\xi} \\ &= \; \mathcal{M}\frac{d}{dt}\left\{ \frac{dU}{dt}\,\hat{e}_1 \; + \; \frac{dV}{dt}\,\hat{e}_2 \; + \; \frac{dW}{dt}\,\hat{e}_3 \right\} \\ & \quad - \; \mathcal{M}\frac{d}{dt}\,\{ U \, \omega_0 \, cos\varphi_0 \, \hat{e}_2 \} \\ & \quad + \; \mathcal{M}\frac{d}{dt}\,\{ V \, \omega_0 \, [\, cos\varphi_0 \, \hat{e}_1 \; + \; sin\varphi_0 \, \hat{e}_3 \,] \} \\ & \quad - \; \mathcal{M}\frac{d}{dt}\,\{ W \, \omega_0 \, sin\varphi_0 \, \hat{e}_2 \} \end{aligned}$$

$$\begin{aligned} \vec{F} = \; & \mathcal{M}\left\{ \frac{d^2U}{dt^2}\,\hat{e}_1 \; + \; \frac{d^2V}{dt^2}\,\hat{e}_2 \; + \; \frac{d^2W}{dt^2}\,\hat{e}_3 \right\} \\ & - \; 2\,\mathcal{M}\left\{ \frac{dU}{dt}\,\omega_0 \, cos\varphi_0 \, \hat{e}_2 \right\} \\ & + \; 2\,\mathcal{M}\left\{ \frac{dV}{dt}\,\omega_0 \, [\, cos\varphi_0 \, \hat{e}_1 \; + \; sin\varphi_0 \, \hat{e}_3 \,] \right\} \\ & - \; 2\,\mathcal{M}\left\{ \frac{dW}{dt}\,\omega_0 \, sin\varphi_0 \, \hat{e}_2 \right\} \\ & - \; \mathcal{M}\,(\omega_0)^2 \, \{ U \, cos\varphi_0 \, + \, W \, sin\varphi_0 \, \} \, [\, cos\varphi_0 \, \hat{e}_1 \; + \; sin\varphi_0 \, \hat{e}_3 \,] \\ & - \; \mathcal{M}\,(\omega_0)^2 \, V \, \hat{e}_2 \end{aligned}$$

Problem (5.)

A bead of mass M is constrained to slide along the frictionless surface of a sphere of radius R_0. There is a potential energy associated with the position of the given by

$$U(\vec{r}) \;=\; M\,A_0\,[\,\ell_1\,x \;+\; \ell_2\,y \;+\; \ell_3\,z\,]$$

(a.) To find the Lagrangian for this system we note spherical coordinate are perfect to use

$$x \;=\; R_0\,cos\phi\,sin\theta \quad , \quad y \;=\; R_0\,sin\phi\,sin\theta \quad , \quad z \;=\; R_0\,cos\theta \quad .$$

so that

$$\begin{aligned} T &= \tfrac{1}{2}\,M\,(R_0)^2\left[(\frac{d\,\theta}{dt})^2 \;+\; sin^2\theta\,(\frac{d\,\phi}{dt})^2\right] \\ U &= M\,A_0\,R_0[\,\ell_1\,cos\phi\,sin\theta \;+\; \ell_2\,sin\phi\,sin\theta \;+\; \ell_3\,cos\theta\,] \\ L &= T \;-\; U \end{aligned}$$

(b.) For the equation of motion of this system via the Euler-Lagrange equations we find

$$\begin{aligned} &\left[\frac{\partial\,L}{\partial(\dot{\theta})}\right] \;=\; M\,(R_0)^2\,(\dot{\theta}) \quad , \quad \left[\frac{\partial\,L}{\partial(\dot{\phi})}\right] \;=\; M\,(R_0)^2\,sin^2\theta\,(\dot{\phi}) \quad , \\ &\left[\frac{\partial\,L}{\partial\theta}\right] \;=\; M\,(R_0)^2\,sin\theta\,cos\theta(\dot{\theta})^2 \\ &\qquad -\,M\,A_0\,R_0[\,\ell_1\,cos\phi\,cos\theta \;+\; \ell_2\,sin\phi\,cos\theta \;-\; \ell_3\,sin\theta\,] \quad , \\ &\left[\frac{\partial\,L}{\partial\phi}\right] \;=\; -\;M\,A_0\,R_0\,sin\theta\,[\;-\;\ell_1\,sin\phi \;+\; \ell_2\,cos\phi\,] \\ &\frac{d}{dt}\left[\frac{\partial\,L}{\partial(\dot{\theta})}\right] \;-\; \frac{\partial\,L}{\partial\theta} \;= \\ &\frac{d}{dt}\left[M\,(R_0)^2(\dot{\theta})\right] \;-\; M\,(R_0)^2\,sin\theta\,cos\theta(\dot{\theta})^2 \\ &\qquad +\; M\,A_0\,R_0[\,\ell_1\,cos\phi\,cos\theta \;+\; \ell_2\,sin\phi\,cos\theta \;-\; \ell_3\,sin\theta\,] \;=\; 0 \quad , \\ &\frac{d}{dt}\left[\frac{\partial\,L}{\partial(\dot{\phi})}\right] \;-\; \frac{\partial\,L}{\partial\phi} \;= \\ &\frac{d}{dt}\left[M\,(R_0)^2\,sin^2\theta\,(\dot{\phi})\right] \;+\; M\,A_0\,R_0\,sin\theta\,[\;-\;\ell_1\,sin\phi \;+\; \ell_2\,cos\phi\,] \;=\; 0 \quad . \end{aligned}$$

(c.) To find the Hamiltonian of this system, we first note.

$$\begin{aligned} p_\theta &= \left[\frac{\partial\,L}{\partial(\dot{\theta})}\right] \;=\; M\,(R_0)^2\,(\dot{\theta}) \quad , \\ p_\phi &= \left[\frac{\partial\,L}{\partial(\dot{\phi})}\right] \;=\; M\,(R_0)^2\,sin^2\theta\,(\dot{\phi}) \quad , \end{aligned}$$

and thus

$$
\begin{aligned}
H = &\; p_\theta\,(\dot{\theta}) \;+\; p_\phi\,(\dot{\phi}) \;-\; L \\
= &\left[\frac{(p_\theta)^2}{2\,M\,(R_0)^2}\right] \;+\; \left[\frac{(p_\phi)^2}{2\,M\,(R_0)^2\,sin^2\theta}\right] \\
&+\; M\,A_0\,R_0[\;\ell_1\,cos\phi\,sin\theta \;+\; \ell_2\,sin\phi\,sin\theta \;+\; \ell_3\,cos\theta\;]
\end{aligned}
$$

Problem (6.)

Given two particles of mass M_1 and M_2 with coordinates $(x_1,\, y_1,\, z_1)$ and $(x_2,\, y_2,\, z_2)$ are constrained to the surface of the same sphere, we can introduce spherical coordinates for both and use angular coordinates for both. The potential energy of the new system is given by

$$
\begin{aligned}
U_{Total} \;=&\; U(\vec{r}_1) \;+\; U(\vec{r}_2) \;+\; \frac{1}{2}k_A\,{R^2}_0\,(6\theta_1 \;-\; 5\theta_2)^2 \\
&+\; \frac{1}{2}k_B\,{R^2}_0\,(\theta_1)^2 \;+\; \frac{1}{2}k_C\,{R^2}_0\,(\theta_2)^2
\end{aligned}
$$

(a.) What is the form of Newton's second law?

$$
\begin{aligned}
\frac{d}{dt}\Big[M\,(R_0)^2(\dot{\theta}_1)\Big] \;=&\; M\,(R_0)^2\,sin\theta_1\,cos\theta_1(\dot{\theta}_1)^2 \\
&-\; M\,A_0\,R_0[\;\ell_1\,cos\phi_1\,cos\theta_1 \;+\; \ell_2\,sin\phi_1\,cos\theta_1 \;-\; \ell_3\,sin\theta_1\;] \\
&-\; 6\,k_A\,{R^2}_0\,(6\theta_1 \;-\; 5\theta_2) \;-\; k_B\,{R^2}_0\,(\theta_1) \\
\frac{d}{dt}\Big[M\,(R_0)^2\,sin^2\theta_1\,(\dot{\phi}_1)\Big] \;=&\; -\;M\,A_0\,R_0\,sin\theta_1\;[\;-\;\ell_1\,sin\phi_1 \;+\; \ell_2\,cos\phi_1\;]\quad. \\
\frac{d}{dt}\Big[M\,(R_0)^2(\dot{\theta}_2)\Big] \;=&\; M\,(R_0)^2\,sin\theta_2\,cos\theta_2(\dot{\theta}_2)^2 \\
&-\; M\,A_0\,R_0[\;\ell_1\,cos\phi_2\,cos\theta_2 \;+\; \ell_2\,sin\phi_2\,cos\theta_2 \;-\; \ell_3\,sin\theta_2\;] \\
&+\; 5\,k_A\,{R^2}_0\,(6\theta_1 \;-\; 5\theta_2) \;-\; k_C\,{R^2}_0\,(\theta_2)\quad, \\
\frac{d}{dt}\Big[M\,(R_0)^2\,sin^2\theta_2\,(\dot{\phi}_2)\Big] \;=&\; -\;M\,A_0\,R_0\,sin\theta_2\;[\;-\;\ell_1\,sin\phi_2 \;+\; \ell_2\,cos\phi_2\;]\quad.
\end{aligned}
$$

(b.) To describe the equation of motion for this including a discussion of normal modes, eigenmodes and eigenfrequencies. it is first important to look at the potential in problem five. This potential implies a force given by

$$\vec{F} \;=\; -\, M\, A_0\, [\, \ell_1\, \hat{x} \;+\; \ell_2\, \hat{y} \;+\; \ell_3\, \hat{z}\,]$$

$$=\; -\, M\, A_0\, |\ell|^2 \left[\frac{[\, \ell_1\, \hat{x} \;+\; \ell_2\, \hat{y} \;+\; \ell_3\, \hat{z}\,]}{\sqrt{(\ell_1)^2 \;+\; (\ell_2)^2 \;+\; (\ell_3)^2}} , \right]$$

$$=\; -\, M\, A_0\, |\ell|^2\, \hat{n}$$

where $|\ell|^2$ is defined by $\sqrt{(\ell_1)^2 \;+\; (\ell_2)^2 \;+\; (\ell_3)^2}$. This is a constant force with magnitude of $M\, A_0\, |\ell|^2$ directed along the direction of $\hat{n}$. But this is exactly like the force of gravity! It follows that the angle μ with which the force meets with the z-axis is given by

$$cos\mu \;=\; \hat{z} \cdot \hat{n} \;=\; \left[\frac{\ell_3}{\sqrt{(\ell_1)^2 \;+\; (\ell_2)^2 \;+\; (\ell_3)^2}} , \right]$$

This implies that we can use generalized coordinates to simplify the problem

$$\beta_1 \;=\; \theta_1 \;-\; \mu \quad , \quad \beta_2 \;=\; \theta_2 \;-\; \mu$$

and the Lagrangian for the system using the new coordinates takes the form

$$L \;=\; \tfrac{1}{2}\, M\, (R_0)^2 \left[(\frac{d\, \beta_1}{dt})^2 \;+\; sin^2(\beta_1 \;+\; \mu)\, (\frac{d\, \phi_1}{dt})^2 \right]$$

$$+\; \tfrac{1}{2}\, M\, (R_0)^2 \left[(\frac{d\, \beta_2}{dt})^2 \;+\; sin^2(\beta_2 \;+\; \mu)\, (\frac{d\, \phi_2}{dt})^2 \right]$$

$$-\; M\, A_0\, R_0\, |\ell|^2\, [\, cos\beta_1 \;+\; cos\beta_2\,]$$

$$-\; \frac{1}{2} k_A\, R^2{}_0\, (6\beta_1 \;-\; 5\beta_2 \;+\; \mu)^2$$

$$-\; \frac{1}{2} k_B\, R^2{}_0\, (\beta_1 \;+\; \mu)^2 \;-\; \frac{1}{2} k_C\, R^2{}_0\, (\beta_2 \;+\; \mu)^2$$

Now the first benefit of the coordinate change is apparent. The potential is independent of ϕ_1 and ϕ_2! To make further progress it is useful to make the small angle approximation.

$$L \;\approx\; \tfrac{1}{2}\, M\, (R_0)^2 \left[(\frac{d\, \beta_1}{dt})^2 \;+\; sin^2(\mu)\, (\frac{d\, \phi_1}{dt})^2 \right]$$

$$+\; \tfrac{1}{2}\, M\, (R_0)^2 \left[(\frac{d\, \beta_2}{dt})^2 \;+\; sin^2(\mu)\, (\frac{d\, \phi_2}{dt})^2 \right]$$

$$-\; M\, A_0\, R_0\, |\ell|^2\, [\, 2 \;-\; \tfrac{1}{2}\, (\beta_1)^2 \;-\; \tfrac{1}{2}\, (\beta_2)^2\,]$$

$$-\; \frac{1}{2} k_A\, R^2{}_0\, (6\beta_1 \;-\; 5\beta_2 \;+\; \mu)^2$$

$$-\; \frac{1}{2} k_B\, R^2{}_0\, (\beta_1 \;+\; \mu)^2 \;-\; \frac{1}{2} k_C\, R^2{}_0\, (\beta_2 \;+\; \mu)^2$$

which makes it clear that only the β-angles are involved in the normal modes. The equations of motion for these takes the form

$$\frac{d}{dt}\left[M\,(R_0)^2\,(\frac{d\,\beta_1}{dt}) \right] \;=\; -\;M\,A_0\,R_0\,|\ell|^2\,\beta_1 \;-\; 6\,k_A\,R^2{}_0\,(6\beta_1 \;-\; 5\beta_2 \,+\, \mu)$$

$$-\;k_B\,R^2{}_0\,(\beta_1 \,+\, \mu)$$

$$\frac{d}{dt}\left[M\,(R_0)^2\,(\frac{d\,\beta_2}{dt}) \right] \;=\; -\;M\,A_0\,R_0\,|\ell|^2\,\beta_2 \;+\; 5\,k_A\,R^2{}_0\,(6\beta_1 \;-\; 5\beta_2 \,+\, \mu)$$

$$-\;k_C\,R^2{}_0\,(\beta_2 \,+\, \mu)$$

or more simply

$$\frac{d}{dt}\left[\frac{d\,\beta_1}{dt} \right] \;=\; -\;\frac{A_0\,|\ell|^2}{R_0}\,\beta_1 \;-\; 6\,\frac{k_A}{M}\,(6\beta_1 \;-\; 5\beta_2 \,+\, \mu)$$

$$-\;\frac{k_B}{M}\,(\beta_1 \,+\, \mu)$$

$$\frac{d}{dt}\left[\frac{d\,\beta_2}{dt} \right] \;=\; -\;\frac{A_0\,|\ell|^2}{R_0}\,\beta_2 \;+\; 5\,\frac{k_A}{M}\,(6\beta_1 \;-\; 5\beta_2 \,+\, \mu)$$

$$-\;\frac{k_C}{M}\,(\beta_2 \,+\, \mu)$$

and after further simplification

$$\frac{d^2\,\beta_1}{dt^2} \;=\; -\;\left[\frac{A_0\,|\ell|^2}{R_0} \,+\, \left(\frac{36\,k_A + k_B}{M} \right) \right]\,\beta_1 \;+\; \frac{30 k_A}{M}\,\beta_2$$

$$-\;\left(\frac{6\,k_A + k_B}{M} \right)\,\frac{k_B}{M}\,\mu$$

$$\frac{d^2\,\beta_2}{dt^2} \;=\; \frac{30 k_A}{M}\,\beta_1 \;-\; \left[\frac{A_0\,|\ell|^2}{R_0} \,+\, \left(\frac{25\,k_A + k_C}{M} \right) \right]\,\beta_1$$

$$-\;\left(\frac{5\,k_A + k_C}{M} \right)\,\frac{k_B}{M}\,\mu$$

The terms that are independent of the β's can be eliminated via a redefinition. Now it is convenient to define three frequencies

$$\Omega_A \;=\; \sqrt{\frac{30 k_A}{M}} \quad ,$$

$$\Omega_{AB} \;=\; \sqrt{\frac{36\,k_A + k_B}{M} \,+\, \frac{A_0\,|\ell|^2}{R_0}} \quad ,$$

$$\Omega_{AC} \;=\; \sqrt{\frac{25\,k_A + k_C}{M} \,+\, \frac{A_0\,|\ell|^2}{R_0}} \quad ,$$

This leads to the eigenvalue condition

$$\Big[(\omega)^2 \;-\; (\Omega_{AB})^2\Big]\,\Big[(\omega)^2 \;-\; (\Omega_{AC})^2\,\Big] \;-\; (\Omega_A)^4 \;=\; 0$$

$$(\omega)^4 \;-\; \Big[(\Omega_{AB})^2 \;+\; (\Omega_{AC})^2\Big]\,(\omega)^2 \;+\; (\Omega_{AB})^2\,(\Omega_{AC})^2 \;-\; (\Omega_A)^4 \;=\; 0$$

$$(\omega)^4 \;-\; b\,(\omega)^2 \;-\; c \;=\; 0$$

$$(\omega)^2 \;=\; \frac{1}{2}\,\Big[b \pm \sqrt{b^2 \;+\; 4\,c}\,\Big]$$

$$(\omega)^2 \;=\; \frac{1}{2}\,\Big[(\Omega_{AB})^2 \;+\; (\Omega_{AC})^2\Big] \pm \frac{1}{2}\sqrt{[\,(\Omega_{AB})^2 \;-\; (\Omega_{AC})^2\,]^2 \;+\; 4\,(\Omega_A)^4}$$

and this is the standard two coupled oscillator system so the eigenvectors are

$$\vec{E}_- \;=\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \vec{E}_+ \;=\; \begin{bmatrix} 1 \\ -\,1 \end{bmatrix}$$