Hamiltonian Mechanics

Hamiltonian Mechanics can be thought of as a powerful extension built on top of Lagrangian Mechanics, but it also stands on its own as a complete theory. Hamiltonian Mechanics is a primary topic for a graduate-level course in dynamics. For a single-semester undergraduate course, we can barely scratch the surface. In fact, most of the advantages of Hamiltonian Mechanics are not apparent when applied to the systems which we study in Phys 410. Nevertheless it is important to gain some initial familiarity with this formalism.

Advantages of Hamiltonian Mechanics

1) It is more closely related to quantum mechanics

2) It allows for powerful geometric interpretations of the time evolution of a mechanical system
   $\Rightarrow$ phase space, Liouville's Theorem.

3) It is particularly well suited to multi-particle systems such as plasmas, gases, particle beams, and non-linear systems.

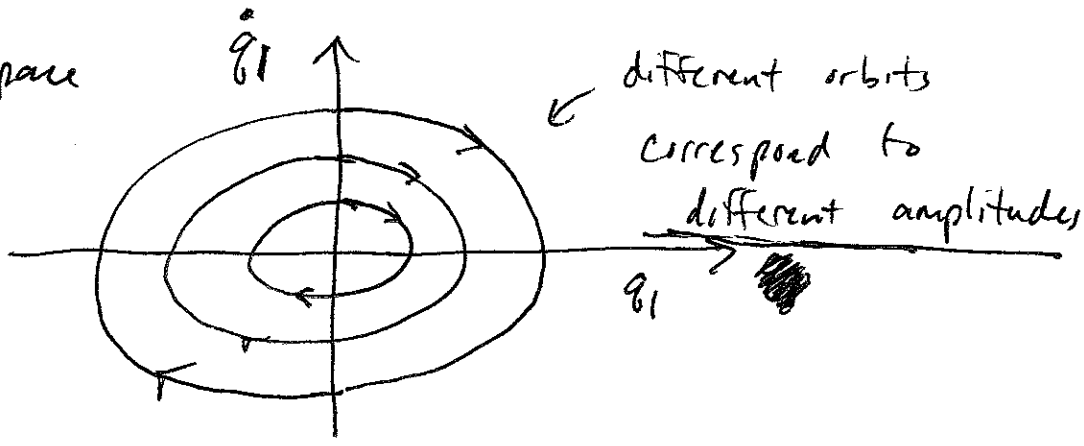
4) It is well suited for applying perturbation Theory.

5) It allows ignorable coordinates to be fully exploited. In fact, Hamilton-Jacobi Theory allows transformations which render all coordinates ignorable.

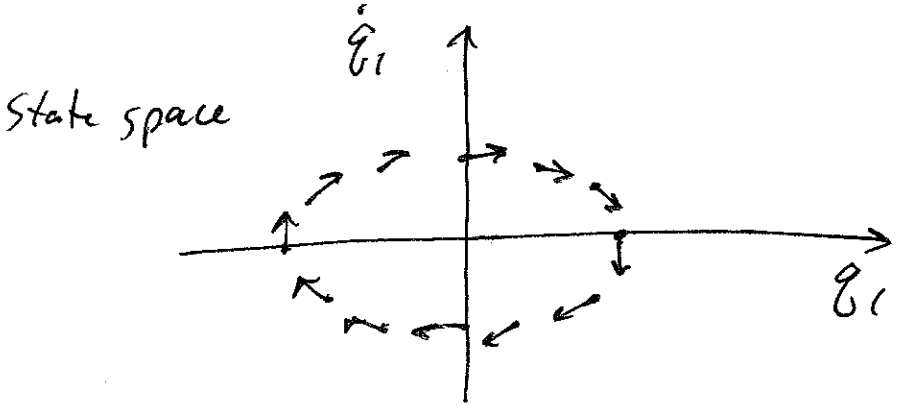
6) The Hamiltonian Equations of Motion are 1st order in time, rather than 2nd order.

State space and phase space.

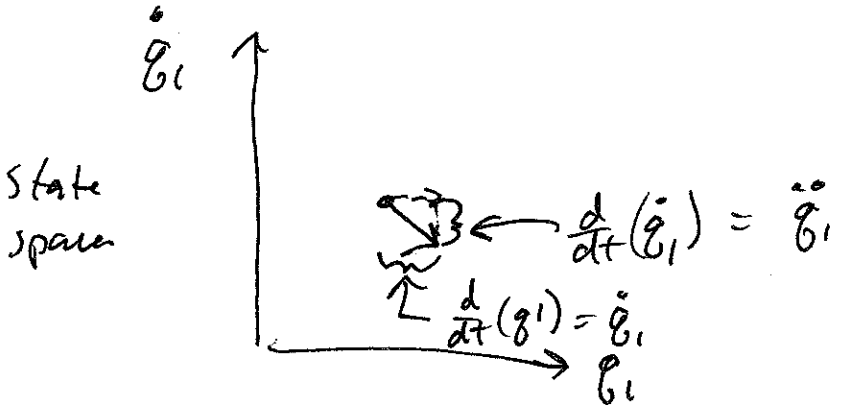
In Lagrangian Theory, the relevant dynamic variables are the generalized coordinates and the generalized velocities: $\{q_i\}$ and $\{\dot{q}_i\}$. These are related by a mathematical operation: $\frac{d}{dt}$. We can think of plotting the evolution of a system by drawing a curve in $(q_i, \dot{q}_i)$ space, called the "state space". For example, a 1D harmonic oscillator would look like



It is useful to think about these curves as being the flow of a vector field, where at each location in the plane we draw a little vector which points in the direction that the system is moving:



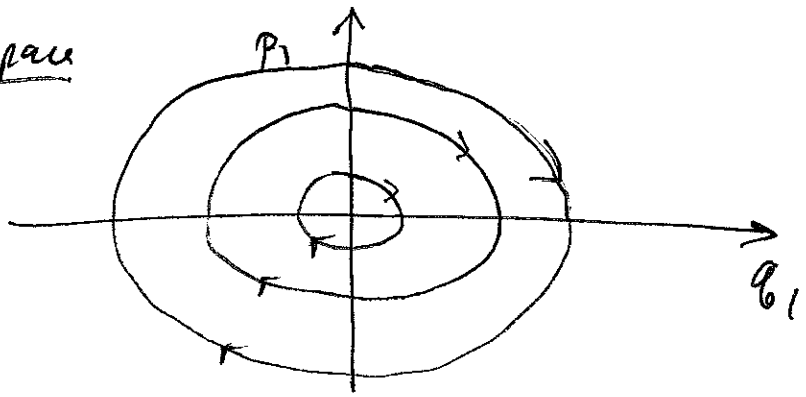
The dynamics of the system, along with the state space that we have chosen to use, should determine how these vector fields should appear. However, Lagrangian mechanics is not well suited for this line of reasoning. The problem is that each vector is the time derivative of q_i and $\dot{q}_i$:



In Lagrangian Mechanics, $\ddot{q}_i$, the "generalized acceleration", is determined only after the entire system is solved. In other words, we only determine $\ddot{q}_i$ indirectly by solving for the entire motion. In Hamiltonian Mechanics, the situation is different, because we use the generalized momentum instead of the generalized velocity, and the time derivative ~~of~~ is ~~explicitly~~ determined explicitly by the equations of motion.

In Hamiltonian Mechanics, we use p_i = generalized momentum, along with q_i. We call this "phase space". For a harmonic oscillator, $p_1 = m\dot{q}_1$, just proportional to the generalized velocity, so the phase space curves looks similiar to the state space curves:

Phase space

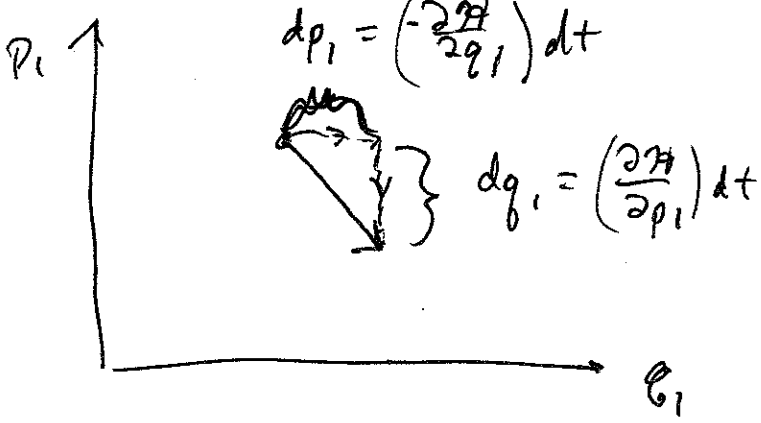


The improvement comes from the equations of motion: in Hamiltonian theory those equations are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

($\mathcal{H}$ is the Hamiltonian)

So the time evolution in phase space is determined directly from the dynamics of the system:



It turns out that this allows for powerful geometric insights into the behavior of the system particularly for complex systems.

The basic procedure.

Usually we begin by finding the Lagrangian. Then from the Lagrangian we determine the generalized momenta:

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Then we get the Hamiltonian, which is defined by

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

, $i = 1, 2, \ldots$, number of degrees of freedom.

For a 1Dimensional single particle it is

$$\mathcal{H} = p\dot{q} - \mathcal{L}$$

Then the equations of motion are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Hamilton Equations of Motion.

Notice that in Lagrangian Mechanics, we get

- one 2nd order Eq. of Motion of each degree of freedom.

While in Hamiltonian Mechanics we get

- 2 1ST order Eqs. of Motion for each degree of freedom.

A simple Example

1 Dimensional particle subject to a potential: $U(x)$.

Then $\mathcal{L}(x, \dot{x}) = T - U = \frac{1}{2}m\dot{x}^2 - U(x)$

Generalized momentum:

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Hamiltonian: $\mathcal{H} = p\dot{x} - \mathcal{L} = p\dot{x} - \left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$

Here we have $\mathcal{H}$ written as a function of $p, \dot{x}$, and x. But our goal in Hamiltonian mechanics is to not have any velocities appearing explicitly. So we will eliminate $\dot{x}$:

$$\dot{x} = \frac{p}{m}$$

So

$$\mathcal{H} = p\left(\frac{p}{m}\right) - \left(\frac{1}{2}m\left(\frac{p}{m}\right)^2 - U(x)\right)$$

$$= \frac{p^2}{m} - \frac{1}{2}\frac{p^2}{m} + U(x)$$

$$\boxed{\mathcal{H} = \frac{1}{2}\frac{p^2}{m} + U(x)}$$

Then the two 1st order Hamiltonian Eqs. of Motion are

$$\dot{x} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$\Downarrow \qquad \Downarrow$$

$$\boxed{\dot{x} = \frac{p}{m}} \qquad \boxed{\dot{p} = -\frac{\partial U}{\partial x} = -\frac{dU}{dx}} = F$$

↑ in Newtonian Mechanics

↑ This equation just returns to us the definition of p.

↑ This equation is the one which determines how p evolves in time.

Another simple example – 2 degrees of freedom

Central Force problem, potential $= U(r)$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2), \quad U(r)$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \quad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$H = p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L}$$

$$= p_r\dot{r} + p_\phi\dot{\phi} - \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)\right]$$

Eliminate all velocities in favor of momenta:

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\phi} = \frac{p_\phi}{mr^2}$$

$$\mathcal{H} = p_r\left(\frac{p_r}{m}\right) + p_\phi\left(\frac{p_\phi}{mr^2}\right) - \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\phi}{mr^2}\right)^2\right) + U(r)$$

$$= \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\phi^2}{mr^2} + U(r)$$

$$\boxed{= \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r)}$$

Four 1st order Eqs. of Motion:

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \Leftarrow \text{just the definition of } p_r$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = +\frac{p_\phi^2}{mr^3} - \frac{dU}{dr} \Leftarrow \text{radial Eq. of Motion.}$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2} \Leftarrow \text{just the definition of } p_\phi$$

$$\dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0 \Leftarrow p_\phi = \text{constant} = \text{angular momentum}$$

Justification for Hamilton's Eqs. of Motion.

$\mathcal{H}$ is a function of p and q, whereas $\mathcal{L}$ is a function of q and $\dot{q}$. So we can write

$$\mathcal{H}(q,p) = p\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))$$

because $\dot{q}$ is a function of q and p

(for example, $\dot{\phi} = \frac{p_\phi}{mr^2} = \dot{\phi}(p_\phi, r)$)

To justify the Hamilton Eqs. of Motion, take the partial derivatives with respect to q and p and apply the Euler-Lagrange Eq:

$$\frac{\partial \mathcal{H}(q,p)}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \left(\frac{\partial \mathcal{L}}{\partial q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial q}\right)$$

$$= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - p\frac{\partial \dot{q}}{\partial q}$$

(cancel)

$$= -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -\frac{d}{dt}(p) = -\dot{p}$$

(by the Euler-Lagrange Eq.)

$$\therefore \boxed{\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}}$$

Similarly,

$$\frac{\partial \mathcal{H}}{\partial p} = \left(\dot{q}_0 + p\frac{\partial \dot{q}}{\partial p}\right) - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p}$$

Cancel

$$= \dot{q}$$

$$\therefore \boxed{\dot{q} = \frac{\partial \mathcal{H}}{\partial p}}$$

Conservation of Energy

Suppose that the Lagrangian is time-independent.

$$\mathcal{L} = \mathcal{L}(q, \dot{q})$$

Then $\frac{\partial \mathcal{L}}{\partial t} = 0$ (because there is no explicit time dependence)

But $\mathcal{L}$ does change in time through the q and $\dot{q}$ variables:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q}\underbrace{\frac{dq}{dt}}_{\dot{q}} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\underbrace{\frac{d\dot{q}}{dt}}_{\ddot{q}}$$

Apply the Euler-Lagrange Equation:

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Then $$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q}\right)\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}\right)$$

or $$\frac{d}{dt}\left(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\dot{q} - \mathcal{L}\right) = 0$$

$$\frac{d}{dt}\underbrace{\left(p\dot{q} - \mathcal{L}\right)}_{\mathcal{H}} = 0 \quad \boxed{\Rightarrow \frac{d}{dt}(\mathcal{H}) = 0 \Rightarrow \mathcal{H} = \text{constant}}$$

This is $\mathcal{H}$, the Hamiltonian.

∴ The Hamiltonian is constant in time if the Lagrangian has no explicit time dependence.

Furthermore, not only is the Hamiltonian constant under these conditions, but also its numerical value has a simple physical interpretation.

To see this, assume that $\mathcal{L}$ does not explicitly depend on the velocity ($\dot{q}$) nor the time.

Then $\frac{\partial U}{\partial \dot{q}} = 0$, so $\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial (T-U)}{\partial \dot{q}}$

$$\Downarrow$$

$$p = \frac{\partial T}{\partial \dot{q}}$$

Then the Hamiltonian can be written

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p\dot{q} - (T-U) = p\dot{q} - T + U$$

$$\uparrow$$

$$p = \frac{\partial T}{\partial \dot{q}}$$

$$= \frac{\partial T}{\partial \dot{q}}\dot{q} - T + U$$

Theorem: $\frac{\partial T}{\partial \dot{q}}\dot{q} = 2T$

Proof: $T = \frac{1}{2}m\dot{x}^2$, $\dot{x} = \frac{\partial x}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial x}{\partial t}$

where $x(q)$ ↑ zero if time does not explicitly appear in the transformation between x & q

$$\dot{x} = \frac{\partial x}{\partial q}\dot{q}, \text{ so } T = \frac{1}{2}m\left(\frac{\partial x}{\partial q}\right)^2(\dot{q})^2$$

$$\frac{\partial T}{\partial \dot{q}} = \frac{1}{2}m\left(\frac{\partial x}{\partial q}\right)^2(2\dot{q}) = m\left(\frac{\partial x}{\partial q}\right)^2\dot{q}$$

and $$\boxed{\dot{q}\frac{\partial T}{\partial \dot{q}} = m\left(\frac{\partial x}{\partial q}\dot{q}\right)^2 = 2T}$$

Finally, the numerical value of $\mathcal{H}$ is

$$\mathcal{H} = \underbrace{\frac{\partial T}{\partial \dot{q}}\dot{q}}_{2T} - T + U = T + U = \text{energy.}$$

So the numerical value of the Hamiltonian is the energy when

1) The equations connecting the (q) and the (x) coordinates are independent of time

2) The potential is velocity independent (and time independent)

We can now compare the Lagrangian to the Hamiltonian:

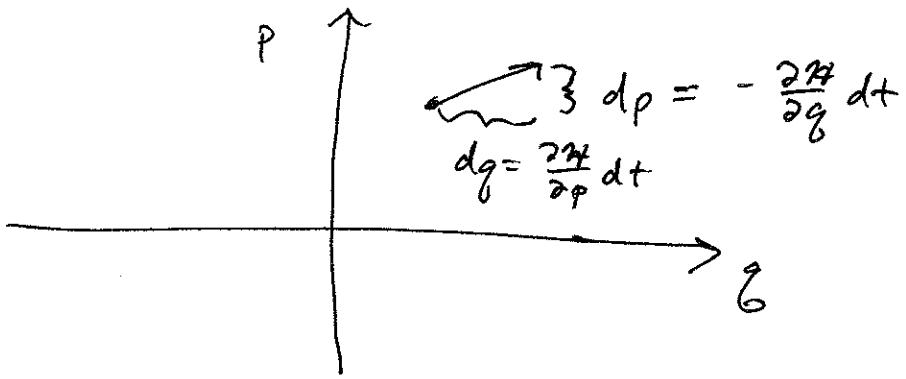
1) Both the Lagrangian and the Hamiltonian are important function which describe the physics of the system. From each of them we can derive the equations of motion for the system.

2) The numerical value of the Lagrangian changes in time and is not very interesting.

$\Rightarrow$ We never ask the question "What is the value of the Lagrangian at time $= t_0$?" We simply don't care about the value of the Lagrangian.

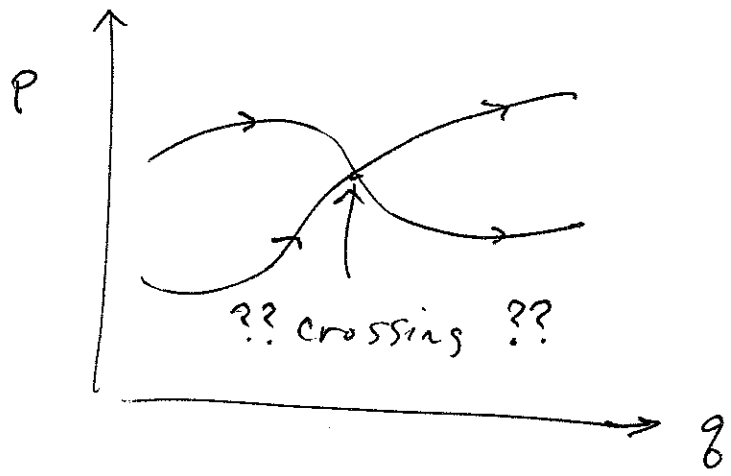
3) In contrast, the numerical value of the Hamiltonian is often the total energy of the system, which is often conserved. So we do care about the value of the Hamiltonian.

4) Nevertheless, you should not think of the Hamiltonian primarily as the energy. You should think of it as the function which governs how the system evolves in time in phase space:

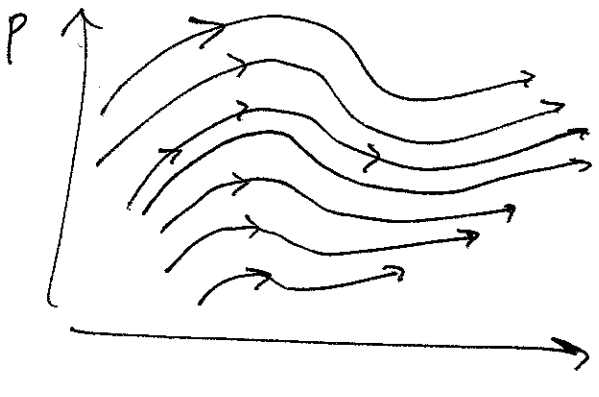


Phase Space Orbits

It is useful to think geometrically about phase space. For example, we can ask the question "can two trajectories in phase space ever cross each other?"



We can see that this cannot happen because the partial derivatives of $\mathcal{H}$ have one and only one value at the crossing point. Therefore both trajectories must evolve in time in the same way from that point onward. Instead we should imagine multiple trajectories as flowing next to each other:



Example: A falling body, subject to gravity.

The Hamiltonian is $\mathcal{H} = T + U = \frac{p^2}{2m} - mgx$

($+x$) points down.

Eqs of Motion:

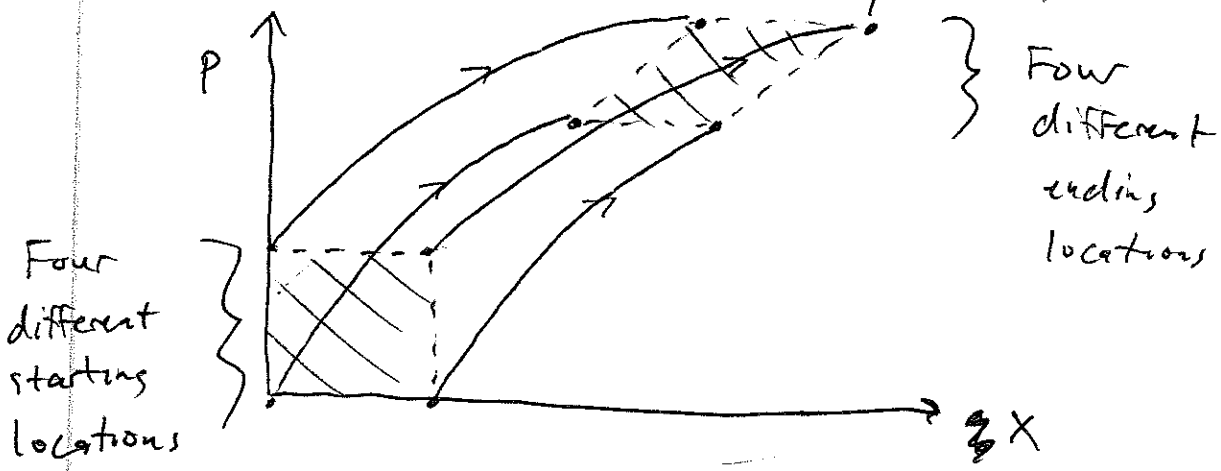
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = mg$$

$$\Rightarrow \boxed{p(t) = p_0 + mgt}$$

Then $\dot{x} = \frac{1}{m}(p_0 + mgt)$

$$\boxed{x(t) = x_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2}$$

What does it look like in phase space?



The initial rectangle has evolved into a parallelogram. But the base of the parallelogram is equal to the base of the rectangle, and its height is equal to the height of the rectangle

The phase space area enclosed by the four trajectories is unchanged.

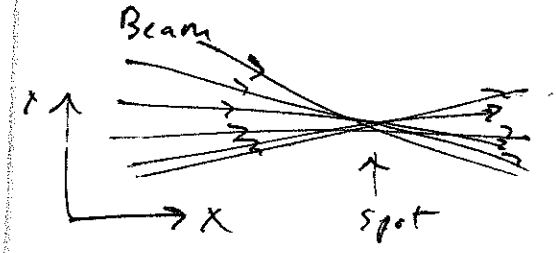
This is an example of Liouville's Theorem

> ⇒ The density of a unit of phase space is unchanged as it evolves in time.

For a many-particle system like an ideal gas, or a plasma, or a large number of stars in a galaxy, this result is a powerful way to analyze the system.

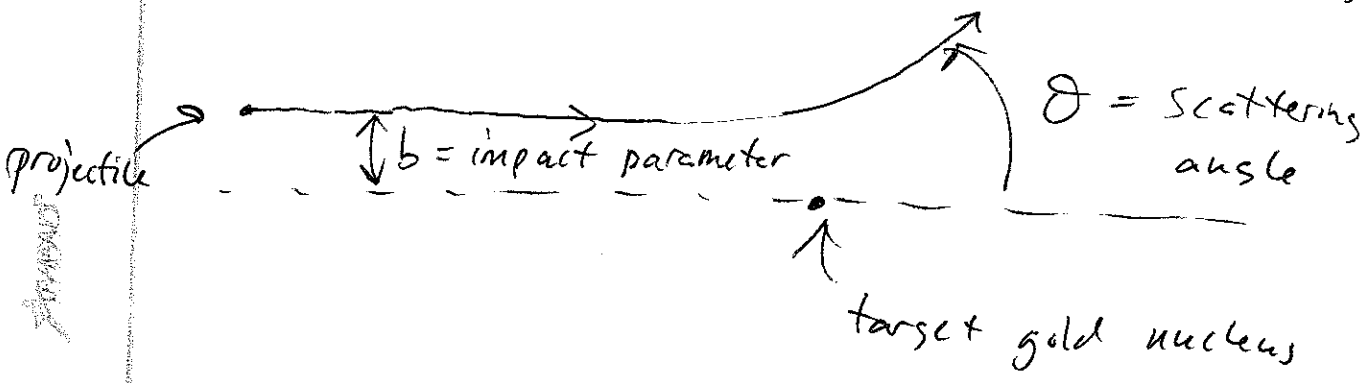
One of the advantages of Hamiltonian Mechanics over Lagrangian Mechanics is that there is no analogous result to Liouville's Theorem in state space.

One simple consequence of Liouville's Theorem is that in a many-particle system, if we wish to see the momentum spread reduced, then we must see the position spread increase. Conversely, for a beam of particles, we can focus them to a small spot, but only by giving them additional momentum spread which will cause an angular divergence:

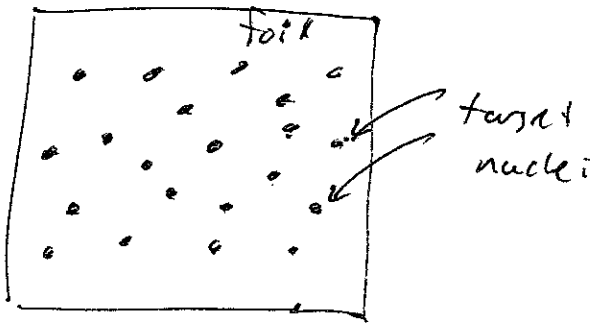


Rutherford Scattering

Suppose we aim a projectile at a small target:



The nucleus is one of many nuclei in a thin foil. The foil is thin enough that an incoming projectile scatters off one and only one target nucleus:



Imagine that ~~the~~ each nucleus acts like a hard billiard ball: the projectile only scatters if it hits the geometric area of the nucleus. Then the number of ~~target~~ projectile which scatter is

$$N_{scatter} = N_{incoming}\left(n_{target}\,\sigma\right)$$

n_{target}: number of targets per unit area

σ: cross sectional area of one nucleus

Units: $N_{scatter}$ and $N_{incoming}$ are unitless integers.

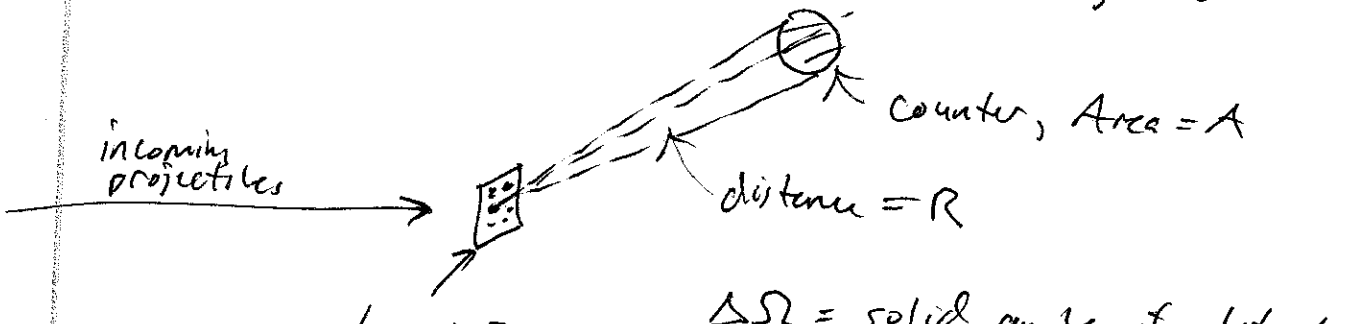
$$n_{target} = \frac{\text{number}}{m^2}$$

σ = cross sectional area = m^2

In practice the projectiles are scattered by the coulomb force, which extends to infinity. So σ is not the physical, geometric area, but it still has units of area. We always call it the total cross-section.

Cross section Units: 1 barn = $10^{-24}\ cm^2 = 10^{-28}\ m^2$

Often we wish to know not just the total scattering cross-section, but also the angular distribution of the scattered projectiles. So we consider a small solid angle $d\Omega$:



$\Delta\Omega$ = solid angle of detector

$$= \frac{A}{R^2}$$

In the limit of $A \to 0$, $\Delta\Omega \to d\Omega$,

and in polar coordinates,

$$dA = r^2 \sin\theta \, d\theta \, d\phi$$

$$d\Omega = \frac{dA}{r^2} = \sin\theta \, d\theta \, d\phi$$

The differential cross-section can be written

$$d\sigma \left(\begin{array}{c} \text{For scattering} \\ \text{into } d\Omega \end{array} \right) = \frac{d\sigma}{d\Omega} \, d\Omega$$

Then the number of projectiles which are scattered is

$$dN_{scattered} = N_{inc} \, n_{target} \, d\sigma$$

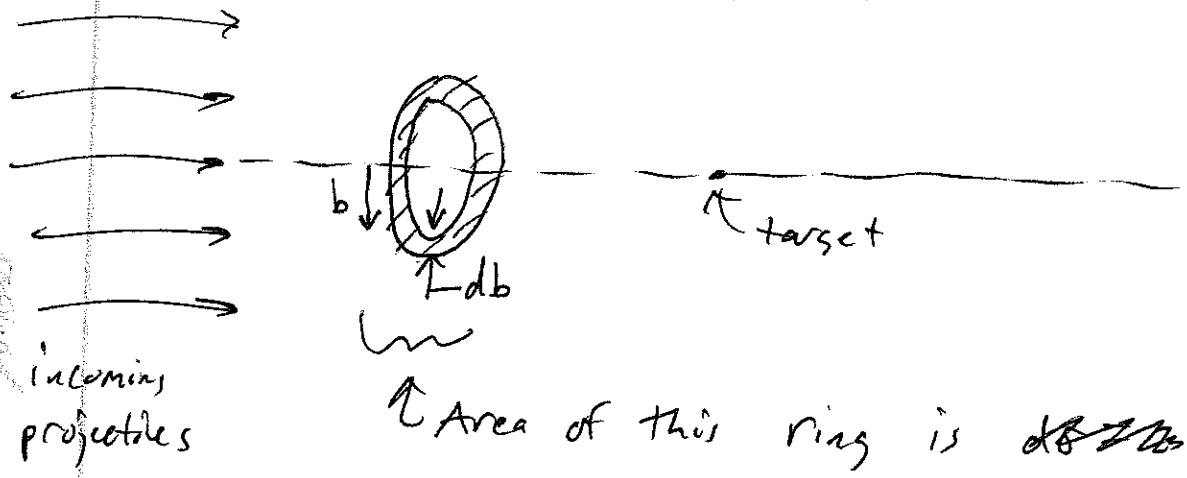
$$= N_{inc} \, n_{target} \, \frac{d\sigma}{d\Omega} \, d\Omega$$

$\frac{d\sigma}{d\Omega}$ = differential cross section, a function of θ & ϕ:

The total cross-section is obtained by integrating the differential cross-section over all solid angles:

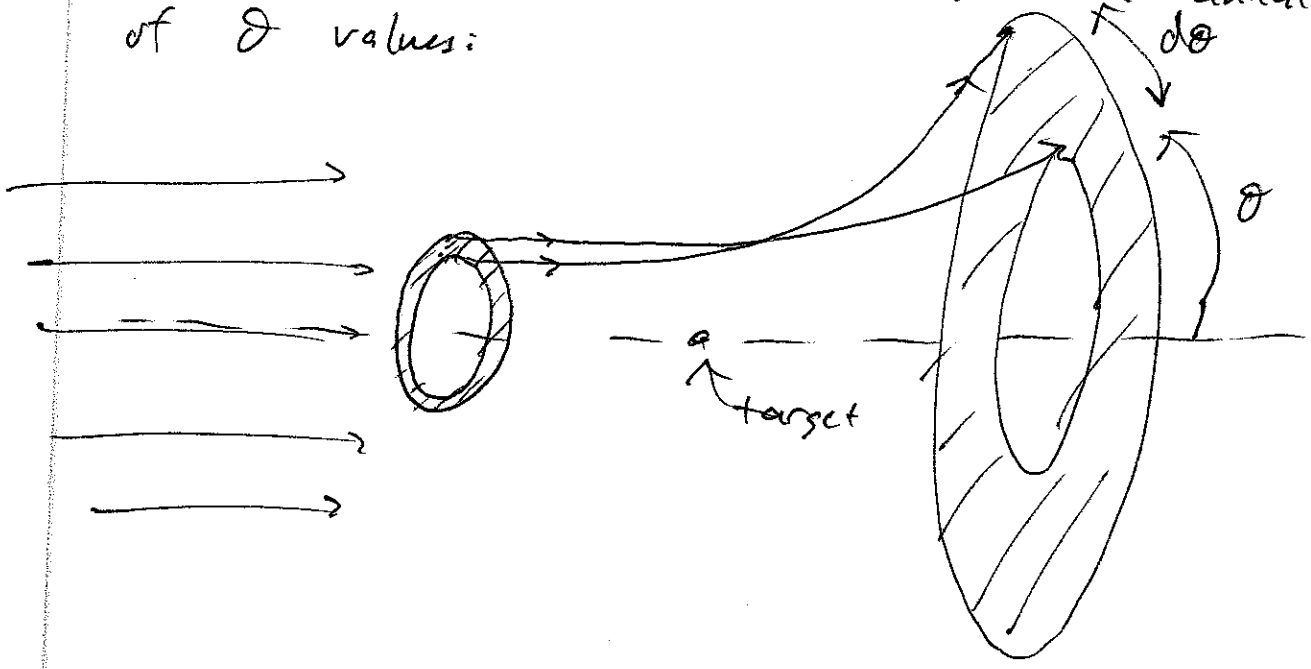
$$\sigma_{total} = \int \frac{d\sigma}{d\Omega} \, d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \frac{d\sigma}{d\Omega}(\theta, \phi)$$

Consider a small range of impact parameters (db), b = impact parameter.



Area of this ring is $(2\pi b)(db)$.

Projectiles which pass within the area of this annulus are scattered into an annulus of θ values:



$$d\Omega = 2\pi \sin\theta \, d\theta$$

So the differential cross section for scattering into this particular $d\Omega$ is the area $(2\pi b)(db)$.

Therefore $$\frac{d\sigma}{d\Omega} = \frac{(2\pi b)\,db}{(2\pi \sin\theta)\,d\theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}$$

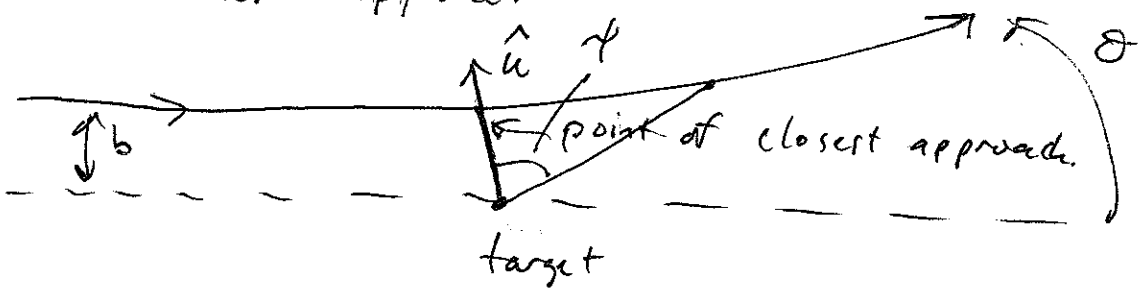
We add an absolute value sign to make sure it is ~~pos~~ positive:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Now we must calculate $\frac{db}{d\theta}$, which describes the relationship between the impact parameter (b) and θ (the scattering angle). We assume that the scattering is due to the coulomb force, and this situation is called Rutherford Scattering.

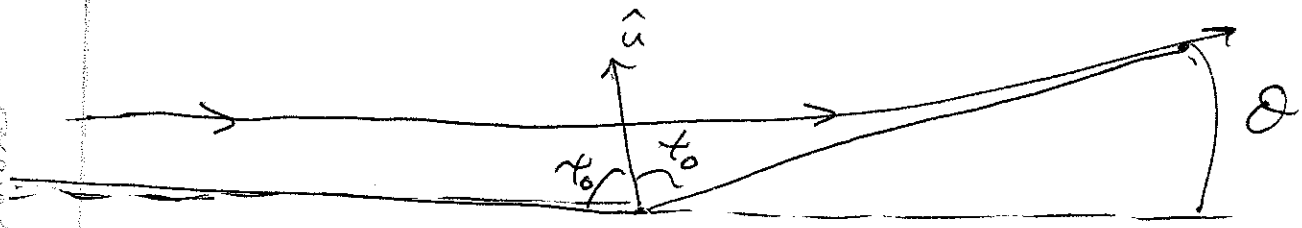
Geometry

The trajectory is symmetric about the point of closest approach



We allow ψ to represent the angle between $\hat{u}$ and the particle's location on the trajectory.

Also let ψ_0 be the asymptotic value for ψ, both for $t = -\infty$ and $t = +\infty$



We can see that $2\psi_0 + \theta = \pi$

or $\theta = \pi - 2\psi_0$.

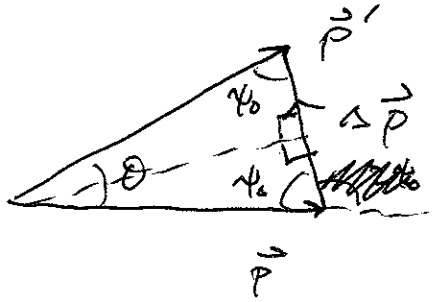
The vector change in momentum,

$$\Delta\vec{p} = \vec{p}\,' - \vec{p}$$

$\vec{p}\,'$: outgoing $\vec{p}$; $\vec{p}$: incoming $\vec{p}$

By conservation of energy, $|\vec{p}\,'| = |\vec{p}|$.

By geometry



, magnitude

$$|\Delta\vec{p}| = 2p\sin\left(\frac{\theta}{2}\right)$$

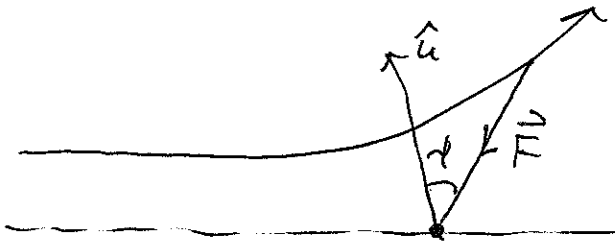
From Newton's 2nd Law,

$$\Delta\vec{p} = \int \vec{F}\,dt$$

Also $\Delta\vec{p}$ is in the direction of $\hat{u}$, so

$$|\Delta\vec{p}| = \int F_u\,dt, \quad F_u = \text{component of } F \text{ in } u \text{ direction}$$

The component of $\vec{F}$ in the u direction is



$$F_u = \frac{\gamma}{r^2}\cos(\psi),$$

$$\gamma = kqQ$$

(q: projectile charge, Q: nuclear charge)

Now $dt = \frac{d\psi}{\dot{\psi}}$

and $\dot{\psi}$ can be found from conservation of angular momentum:

$$mr^2\dot{\psi} = bp$$

(bp: incoming momentum; ℓ: initial angular momentum)

so $\dot{\psi} = \frac{bp}{mr^2}$

Therefore $|\Delta\vec{p}| = \int \frac{\gamma\cos(\psi)}{r^2}\frac{d\psi}{\dot{\psi}}$ ($dt = \frac{d\psi}{\dot{\psi}}$)

$$= \int_{-\psi_0}^{\psi_0} \frac{\gamma\cos(\psi)}{r^2}\frac{d\psi}{(bp/mr^2)}$$

$$= \frac{\gamma m}{bp}\int_{-\psi_0}^{\psi_0}\cos(\psi)\,d\psi$$

$$= \frac{\gamma m}{bp}\,2\sin(\psi_0)$$

$\Sigma\ \psi_0 = \frac{1}{2}(\pi-\theta)$

$$= \frac{2\pi m}{bp}\cos\left(\frac{\theta}{2}\right)$$

Now we have 2 results for $|\Delta\vec{p}|$:

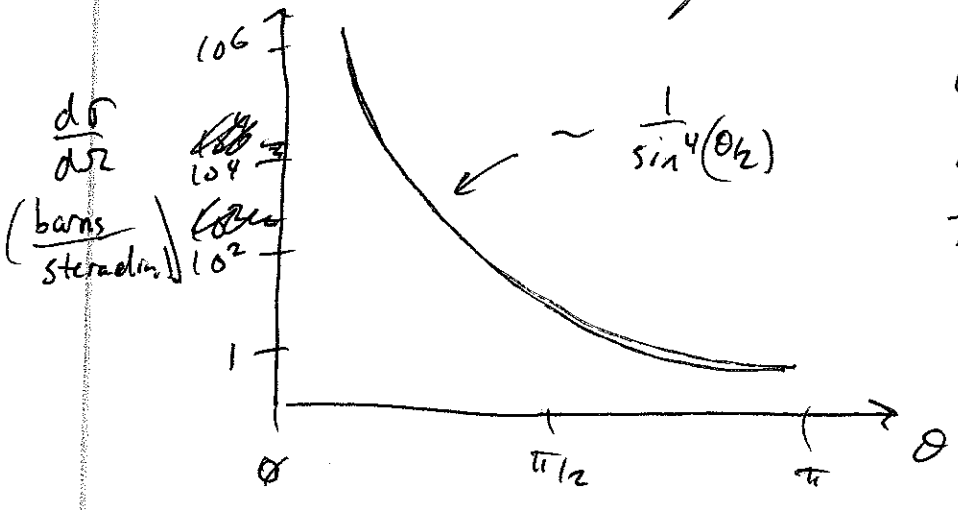
$$|\Delta\vec{p}| = 2p\sin\left(\frac{\theta}{2}\right)$$

and $$|\Delta\vec{p}| = \frac{2\pi m}{bp}\cos\left(\frac{\theta}{2}\right)$$

Equate them and solve for b:

$$b(\theta) = b = \frac{\pi m}{p^2}\frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{\pi}{mv^2}\cot\left(\frac{\theta}{2}\right)$$

Finally, $$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\, b\left|\frac{db}{d\theta}\right| = \left(\frac{kqQ}{4E\sin^2(\theta/2)}\right)^2$$



Geiger and Marsden confirmed the θ dependence and the dependences on $\frac{1}{E^2}$ and Q^2.