Motion of a charge in a uniform magnetic field

$$\vec{F} = q\vec{v} \times \vec{B} = m\dot{\vec{v}}$$

Let $\vec{B}$ be constant and in the $\hat{z}$ direction:

$$\vec{B} = (\emptyset, \emptyset, B)$$

Also $\vec{v} = (v_x, v_y, v_z)$

Then $\vec{v} \times \vec{B} = (v_y B, -v_x B, \emptyset)$

So the three components of $\vec{F} = m\vec{a}$ are

$$m\dot{v}_x = qBv_y$$
$$m\dot{v}_y = -qBv_x$$
$$m\dot{v}_z = \emptyset \implies \boxed{v_z = \text{constant} = v_{z0}.}$$

Simplify x & y equations by defining

$$\omega = \text{"cyclotron frequency"} \equiv \frac{qB}{m}$$

Then
$$\boxed{\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \end{aligned}}$$
Coupled differential Equations

These can be easily solved using a trick with complex numbers.

Define $\eta \equiv v_x + i v_y$

Then $\dot{\eta} = \dot{v}_x + i \dot{v}_y$

$= \omega v_y + i(-\omega v_x)$ by the equations of motion.

$= -i\omega(i v_y + v_x)$

$= -i\omega(\underbrace{v_x + i v_y})$

$= -i\omega\eta$

$$\boxed{\dot{\eta} = -i\omega\eta} \quad \text{Equations of Motion}$$

Solution: $\boxed{\eta = A e^{-i\omega t}}$

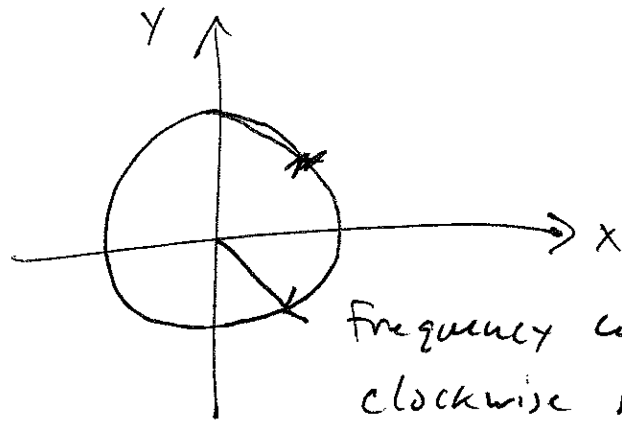
We can integrate once more to get $x(t)$ & $y(t)$:

Define: $\xi = x + i y$. Then $\dot{\xi} = \eta$, so

$$\xi = \int \eta \, dt = \int A e^{-i\omega t} dt$$

$$= \frac{iA}{\omega} e^{-i\omega t} + \left(\begin{array}{c}\text{complex}\\ \text{constant}\end{array}\right)$$

So ξ rotates in the xy plane with frequency ω:



frequency ω,
clockwise motion for $+q$
because $\xi \sim e^{-i\omega t}$

$(-)$ sign means clockwise.

Meanwhile z increases uniformly:

$$\boxed{z(t) = z_0 + v_{0z} t} \Rightarrow \underline{\text{helical motion}}$$

The complex constant of integration can be chosen to be zero by choosing $(x, y) = (0, 0)$ to be the center of the circular motion.

The initial velocities v_{x0} and v_{y0} affect the magnitude and phase of A as well as the center of the motion.

Radius of the orbit $= r = \frac{v}{\omega} = \frac{v}{qB/m} = \frac{mv}{qB} = \frac{p}{qB}$

Conservation of Momentum

By Newton's 3rd law we can infer that

$$\dot{\vec{P}}_{total} = \vec{F}^{ext}$$

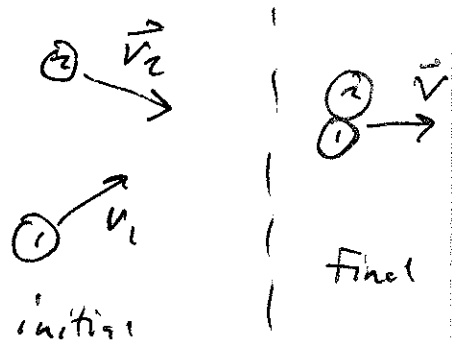
because all internal force cancel.

If $\vec{F}^{ext} = \emptyset$, then total momentum P_{total} is conserved: $\vec{P}_{total} = \text{constant}$

~~Ex~~ Ex: Inelastic collision

Let 2 particles collide and stick together:

$$\vec{P}_{initial} = m_1\vec{v}_1 + m_2\vec{v}_2$$



$$\vec{P}_{final} = \underbrace{m_1\vec{v} + m_2\vec{v}}_{\text{common final velocity}} = (m_1+m_2)\vec{v}$$

so $\vec{P}_{initial} = \vec{P}_{final}$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}$$

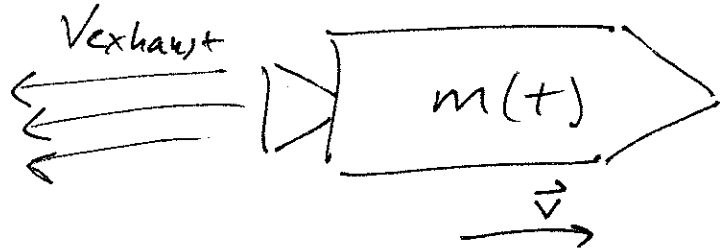
$$\boxed{\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}}$$

Special case: $\vec{v}_2 = \phi$. Then

$$\vec{v} = \left(\frac{m_1}{m_1+m_2}\right)\vec{v}_1$$

Rocket Motion

Rockets in free space accelerate by jettisoning mass. Conservation of momentum gives them a boost. (The jettisoned mass is in the form of exhaust gas.)



For this case the mass of the rocket depends on time:

$$m(t)$$

And $\dot{m}(t)$ is negative, because the rocket is becoming lighter.

At time t, $P(t) = m(t)\,v(t)$

At time $t+dt$, $P(t+dt) = \big(m(t) + dm\big)\big(v(t) + dv\big)$

change in mass, dm is negative.

The exhaust ejected in time dt has mass (dm) and velocity $v(t) - V_{ex}$ relative to an ~~at~~ inertial coordinate system at rest.

So the total momentum is

$$P(t+dt) = \underbrace{(m(t) + dm)(v(t) + dv)}_{\text{rocket}} - \underbrace{dm\,(v(t) - V_{ex})}_{\text{exhaust}}$$

$$= m(t)v(t) + m(t)dv + \cancel{v(t)dm} + dm\,dv$$
$$- \cancel{v(t)dm} + V_{ex}dm$$

ignore for being very small

$$= \underbrace{m(t)v(t)}_{P(t)} + m(t)dv + V_{ex}dm$$

So $dP = P(t+dt) - P(t) = m(t)dv + V_{ex}dm$

Since momentum is conserved, $dP = \emptyset$ ∴

$$m(t)\,dv + V_{ex}\,dm = \emptyset$$
$$m(t)\,dv = -dm\,V_{ex}$$

or
$$m(t)\frac{dv}{dt} = -\frac{dm}{dt}V_{ex}$$

$$\boxed{m(t)\dot{v} = -\dot{m}(t)V_{ex}}$$

Eq. of Motion

This looks like an ordinary Newton's 2nd Law Eq. of Motion where $-\dot{m}(t)V_{ex}$ plays the role of the force. We call it the thrust.

$$\boxed{\text{Thrust} \equiv -\dot{m}(t)\, v_{ex}}$$

We can solve the Eq. of motion by seperation of variables:

$$dv = \frac{-v_{ex}\, dm}{m(t)}$$

$$v_{\text{Final}}^{(t)} - v_0 = -v_{ex} \int_{m_0}^{m(t)} \frac{dm'}{m'(t)}$$

$$= -v_{ex} \ln\left(\frac{m(t)}{m_0}\right)$$

$$\boxed{v_{\text{Final}}^{(t)} - v_0 = v_{ex} \ln\left(\frac{m_0}{m(t)}\right)}$$

Center-of-Mass

we define $\vec{R} \equiv$ center of mass of a system of particles

$$\equiv \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_N \vec{r}_N}{M}$$

where $M \equiv \sum_\alpha m_\alpha$ = total mass.

Three components: $R_x = \frac{1}{M} \sum_\alpha m_\alpha r_x$, $R_z = \frac{1}{M} \sum_\alpha m_\alpha r_z$

$R_y = \frac{1}{M} \sum_\alpha m_\alpha r_y$, The total momentum can be written in terms of the CM:

$$\vec{P}_{total} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

Taking the derivative we have

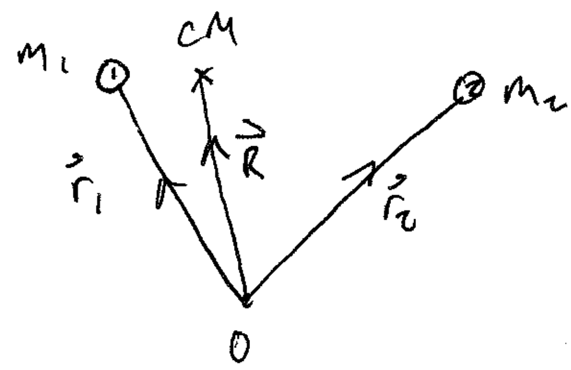
$$\dot{\vec{P}}_{total} = M\ddot{\vec{R}}$$

↑

$\vec{F}^{ext}$ by Newton's 2nd Law

$$\boxed{\vec{F}^{ext} = M\ddot{\vec{R}}}$$

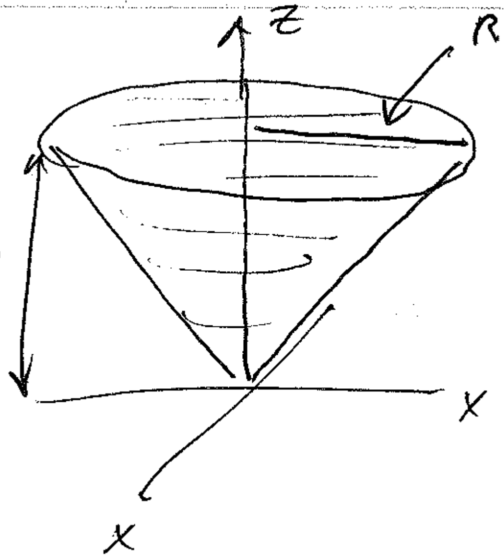
The CM moves like a single particle subject to a single force ($\vec{F}^{ext}$).



For a continuous mass distribution,

$$\vec{R} \equiv \frac{1}{M}\int_{volume} \rho \vec{r}\, dV \quad , \text{ where } \rho(x,y,z) \text{ is the mass density.}$$

<u>Ex</u>: CM of a cone:



uniform solid cone
(density $= \rho =$ constant)

$$R_x = \frac{1}{M}\int \rho(x,y,z)\, x \, dx\,dy\,dz$$

By symmetry, every point
$(\rho(x,y,z)\,x)$
is cancelled by a point $(\rho(-x,y,z)(-x))$

so $R_x = 0$.

Similarly for R_y: $R_y = \emptyset$

For R_z, there is no cancellation:

$$R_z = \frac{\rho}{M}\int z\, dx\,dy\,dz$$

The integrals over $dx\,dy$ give the area of a circle: πr^2.

But r is related to z: $r = \frac{Rz}{h}$

$$\text{So } R_z = \frac{\rho}{M}\int z\, \pi\left(\frac{Rz}{h}\right)^2 dz$$

$$= \frac{\rho R^2 \pi}{M h^2}\int_\emptyset^h z^3\, dz = \frac{\rho R^2\pi}{M h^2}\left(\frac{1}{4}h^4\right)$$

The volume of the cone is

$$V = \frac{1}{3}\pi R^2 h$$

so the mass is $M = \dfrac{\rho \pi R^2 h}{3}$

Therefore $\boxed{R_Z = \frac{3}{4}h}$

Angular Momentum (Single Particle)

We define $\vec{\ell} \equiv \vec{r} \times \vec{p}$, $\vec{r}$ = position vector

$\vec{p}$ = momentum.

Its time rate change is

$$\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}}$$

$\dot{\vec{p}} = \sum \vec{F}^{net}$ by Newton's 2nd Law

$\vec{p} = m\dot{\vec{r}}$, so $\dot{\vec{r}} \times m\dot{\vec{r}} = \emptyset$ (parallel vectors have zero cross product.)

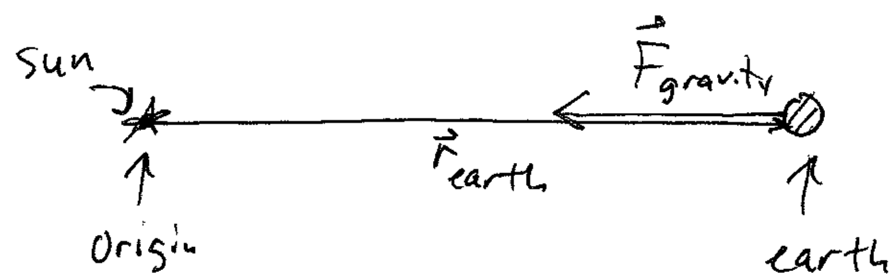
$$\therefore \dot{\vec{\ell}} = \vec{r}\times\vec{F}$$

Define $\vec{r}\times\vec{F} \equiv \vec{\Gamma}$ = "torque"

Then $\boxed{\vec{\Gamma} = \dot{\vec{\ell}}}$ Newton's 2nd Law in ~~ang~~ angular form.

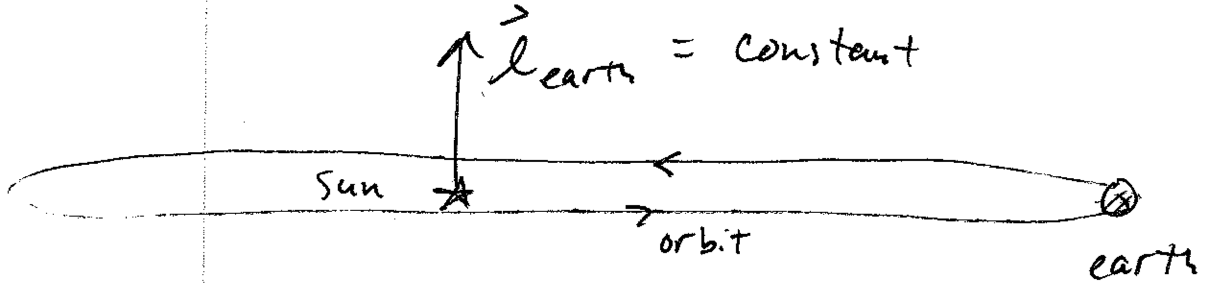
Torque and Angular Momentum ~~are~~ behave somewhat differently than Force and linear momentum. For example, changing the origin changes the torque:

Ex: Central Force (sun-earth system)

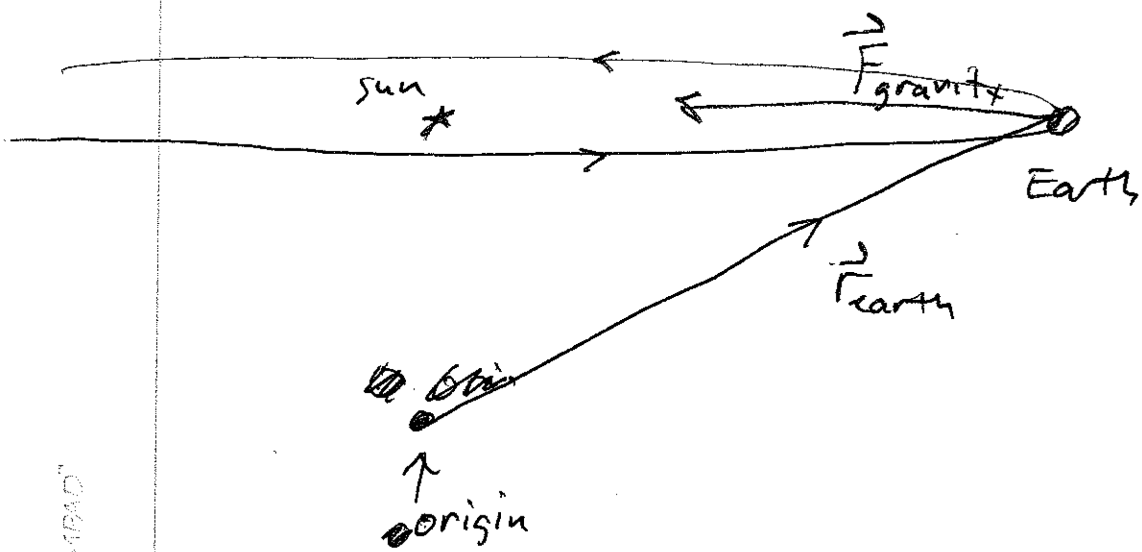


With the sun as the origin,

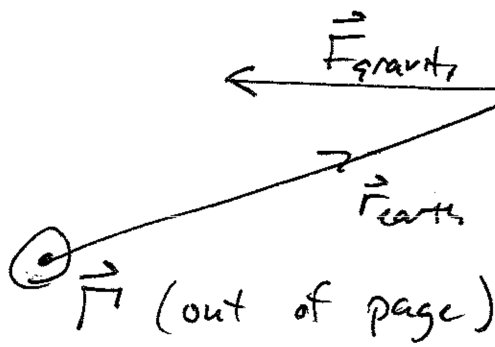
$$\vec{\Gamma} = \vec{r} \times \vec{F}_{gravity} = \emptyset$$ because $\vec{r}$ and $\vec{F}$ are parallel. With this choice of origin, angular momentum is conserved:



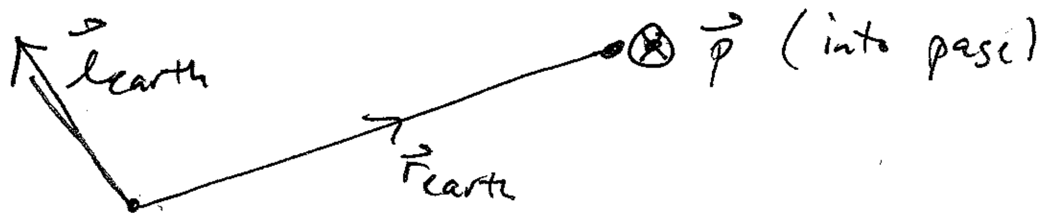
But suppose we choose an origin below the sun, and below the plane of the earth's motion:



Now the torque is non-zero and points out of the page:



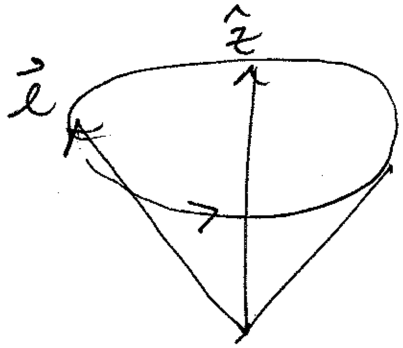
The angular momentum vector is



According to Newton's 2nd Law,

$\vec{\Gamma} = \dot{\vec{l}}$, so $\vec{l}$ must be <u>changing</u> in response to this torque

How is $\vec{l}$ changing? It is rotating around the $\hat{z}$ axis on a cone:



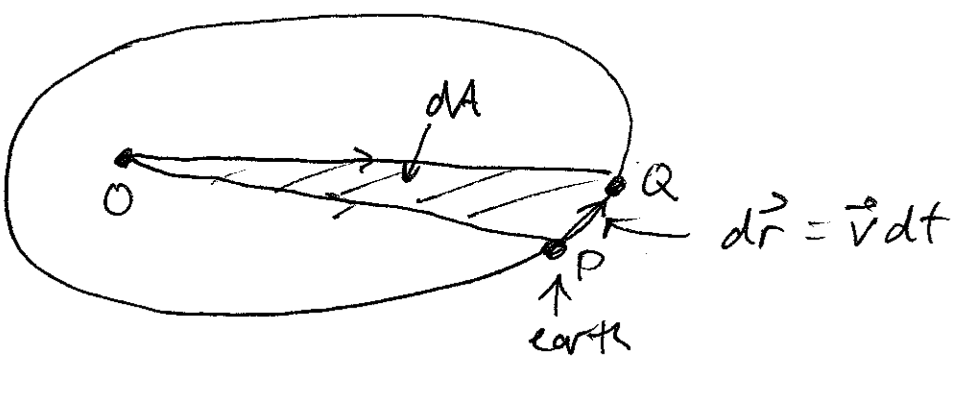
$\vec{l}$ always remains $\perp$ to both $\vec{r}$ and $\vec{p}$.

So with the origin at the sun, the situation is simple: $\vec{\Gamma} = \emptyset$, and $\vec{l}$ = constant. But a different choice of origin gives $\vec{\Gamma}$ = non-zero and $\vec{l}$ changing in time.

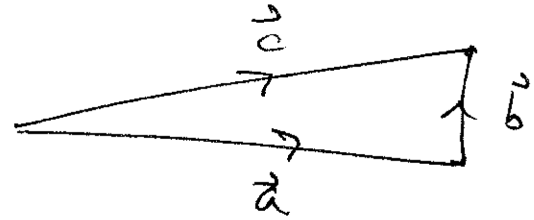
Kepler's 2^{nd} Law:

"As each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal area in equal time."

This is a consequence of angular momentum conservation (taking the sun as the origin.) We can see this as follows.



From homework # 1.18, we know that the area of a triangle can be calculated by taking the cross product of any two of its sides.



$$\text{Area} = \tfrac{1}{2} |\vec{a} \times \vec{b}| \quad \text{for example.}$$

So the area OPQ is

$$dA = \tfrac{1}{2} |\vec{r} \times \vec{v}\, dt|$$

replace $\vec{v}$ by $\vec{p}/m$ and divide by dt:

$$\frac{dA}{dt} = \frac{1}{2m} |\vec{r} \times \vec{p}| = \frac{1}{2m} |\vec{\ell}|$$

But with our choice of origin, $|\vec{\ell}| = \text{constant}$, so

$$\boxed{\frac{dA}{dt} = \text{constant}}$$ This is Kepler's 2nd Law.

Alternatively, $\ell = mr^2\omega$. The geometrical rate is $\dfrac{dA}{dt} = \tfrac{1}{2} r^2 \omega$

So $\dfrac{dA}{dt} = \text{constant}$ if $\ell = \text{constant}$.

Total Angular Momentum of a system of particles:

Define $\vec{L} \equiv \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha$

It is a simple ~~exer~~ exercise to show that if all internal forces between particles obey Newton's 3rd Law (equal & opposite), then

$$\boxed{\dot{\vec{L}} = \vec{\Gamma}^{ext}}$$

where $\vec{\Gamma}^{ext}$ ~~are~~ is the total torque due to external forces. In particular, if $\vec{\Gamma}^{ext} = \phi$, then total angular momentum is conserved.

Moment of Inertia

For a fixed axis, which is not allowed to wobble, the moment of inertia is defined as:

$$I \equiv \sum_\alpha m_\alpha \rho_\alpha^2 ,$$

where $\rho_\alpha \equiv$ distance from particle (α) to the axis.

For a continuous mass distribution,

$$I \equiv \int \rho\, \rho^2 \, dV$$

ρ = distance to axis of rotation.

$\rho = \rho(x,y,z)$ = density function

Some simple items:

~~I uniform disk~~ uniform disk: $I = \frac{1}{2} M R^2$

radius

total mass

solid sphere rotating about one axis: $I = \frac{2}{5} M R^2$

The usefulness of the moment of inertia is that you can quickly calculate the $\hat{z}$-component of $\vec{L}$ (for a fixed rotation axis in the $\hat{z}$ direction) by multiplying

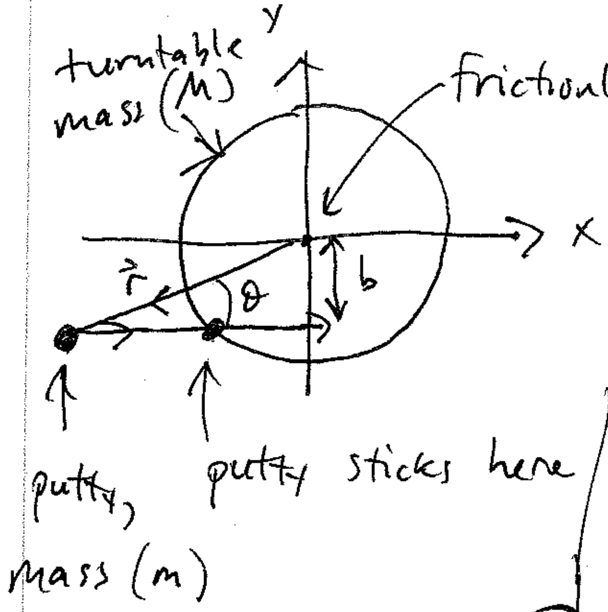
$$L_z = I\omega$$

One can then determine the time rate change by applying Newton's 2nd Law:

$$I\dot{\omega} = \Gamma_z$$

Angular collision problems can be solved ~~to~~ by applying conservation of L.

Ex: Putty thrown at a ~~sta~~ stationary turn table:



Initial Angular Momentum:

$$L_z = r(mv)\sin\theta = mvb$$

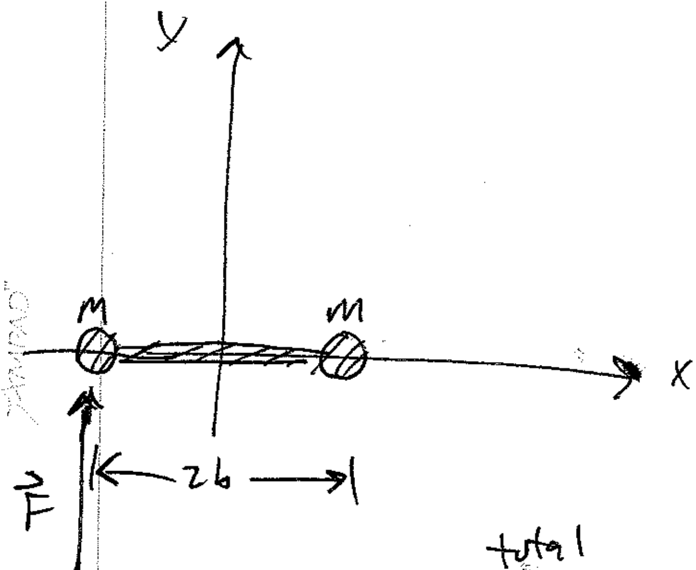
After collision, the moment of inertia is

$$I = \underbrace{mR^2}_{\text{I for the putty}} + \underbrace{\frac{M}{2}R^2}_{\text{I for the turntable}} = \left(m + \frac{M}{2}\right)R^2$$

$$L_z^{\text{final}} = \left(m + \frac{M}{2}\right)R^2\omega = mvb$$

$$\boxed{\omega = \frac{m}{\left(m + \frac{M}{2}\right)} \frac{vb}{R^2}}$$

Ex: Struck Dumbell



A force strikes the left dumbbell for a short time Δt.

$M = 2m$

The change in total linear momentum is

$$\Delta \vec{P} = \vec{F} \Delta t$$

Starting with $\vec{P}_{initial} = \emptyset$, we have

$$\vec{P}_{final} = \underset{\substack{\uparrow \\ \text{total mass}}}{M} \underset{\substack{\uparrow \\ \text{velocity of CM}}}{\dot{\vec{R}}} = \vec{F} \Delta t \implies \boxed{\dot{\vec{R}} = \vec{v}_{cm} = \left(\frac{\vec{F}\Delta t}{M}\right)}$$

The change in total angular momentum about the origin is due to the torque

$$|\Gamma^{ext}| = Fb$$

$$\Delta L = |\Gamma^{ext}| \Delta t = Fb(\Delta t)$$

$L_{initial} = \emptyset$, so $\Delta L = L_{final} = I\omega$

$$I = 2mb^2 = Mb^2, \text{ so}$$

$$I\omega = Fb\Delta t$$

$$(Mb^2)\omega = Fb\Delta t$$

$$\boxed{\omega = \frac{F\Delta t}{Mb} = \text{angular velocity}}$$

The left mass has velocity:

$$v_{left} = v_{cm} + \omega b = \frac{F(\Delta t)}{M} + \left(\frac{F(\Delta t)}{Mb}\right)b$$

$$= \frac{2F(\Delta t)}{M}$$

$$= \frac{\cancel{2}F(\Delta t)}{M/2}$$

$$\boxed{= \frac{\cancel{2}F\Delta t}{m}}$$

The right mass initially has velocity

$$\boxed{v_{right} = v_{cm} - \omega b = \emptyset}$$

The CM moves vertically up the y-axis while the dumbbell rotates.

Work & Energy

We define $KE \equiv T \equiv \frac{1}{2}mv^2$ for a single particle.

The time derivative is

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right)$$

$$= \frac{1}{2}m\dot{\vec{v}}\cdot\vec{v} + \frac{1}{2}m\vec{v}\cdot\dot{\vec{v}}$$

$$\frac{dT}{dt} = m\dot{\vec{v}}\cdot\vec{v}$$

From Newton's 2nd Law, $m\dot{\vec{v}} = \dot{\vec{p}} = \vec{F}^{ext}$, so

$$\frac{dT}{dt} = \vec{F}\cdot\vec{v} = \vec{F}\cdot\frac{d\vec{r}}{dt}$$

or $$\boxed{dT = \vec{F}\cdot d\vec{r}}$$

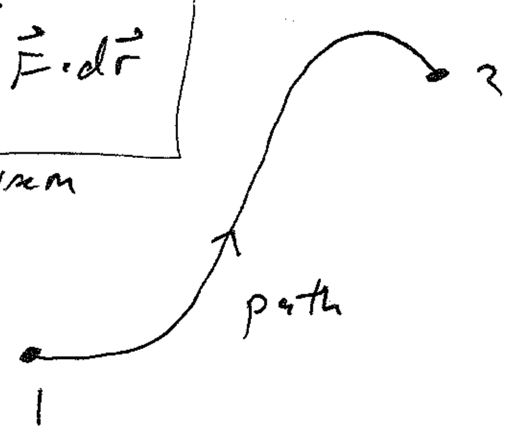
We define $\vec{F}\cdot d\vec{r}$ to be the <u>differential work</u> (dW) done by force $\vec{F}$:

$$dW \equiv \vec{F}\cdot d\vec{r}$$

The total work done is found by integrating dW over the entire path:

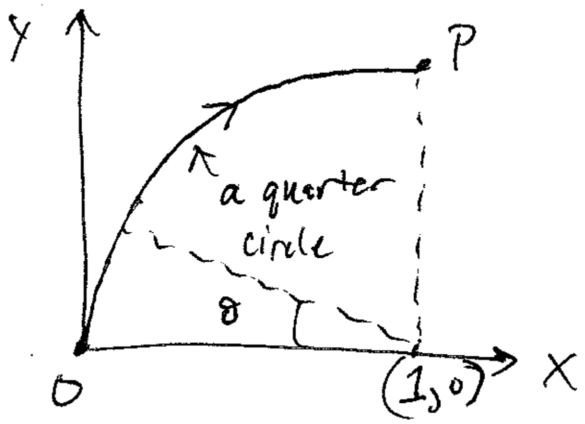
$$\boxed{\Delta T = T_2 - T_1 = \int_1^2 \vec{F}\cdot d\vec{r}}$$

Work-energy theorem



The Work ~~Energy~~ Energy Theorem follows from Newton's 2^{nd} Law and our definitions of F, T & W.

Example Work calculation:



Path: $\vec{r} = (1-\cos\theta, \sin\theta)$

$d\vec{r} = (\sin\theta, \cos\theta)\, d\theta$

Force Law:

$\vec{F} = (y, 2x)$

$= y\hat{x} + 2x\hat{y}$

Notice that $d\vec{r} = \underbrace{(\sin\theta\,\hat{x} + \cos\theta\,\hat{y})}_{\text{direction}}\, \underset{\text{magnitude}}{d\theta}$

Line integral along path:

$$W = \int_0^P \underbrace{(y\hat{x} + 2x\hat{y})}_{\downarrow} \cdot (\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\, d\theta$$

$$= \int_\phi^{\pi/2} (\sin\theta\,\hat{x} + 2(1-\cos\theta)\,\hat{y}) \cdot (\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\, d\theta$$

$$= \int_\phi^{\pi/2} (\sin^2\theta + 2(1-\cos\theta)\cos\theta)\, d\theta = \boxed{2 - \pi/4}$$

Starting with $\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$, show that

$$T_2 - T_1 = \Delta T = \int_1^2 \vec{F} \cdot d\vec{r}$$

Proof. Integrate both sides over time:

$$\int_1^2 \frac{dT}{dt} \, dt = \int_1^2 \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt$$

Left hand side is $T_2 - T_1 = \Delta T$

Right hand side is $\lim_{\Delta t \to 0} \sum_i \vec{F}_i \cdot \frac{\Delta \vec{r}_i}{\Delta t_i} \Delta t_i$

$$= \lim_{\Delta \vec{r} \to 0} \sum_i \vec{F}_i \cdot \Delta \vec{r}_i$$

$$= \int_1^2 \vec{F} \cdot d\vec{r}$$

This argument is true no matter what $\vec{F}$ depends upon, including time, position, and velocity.

Specifically, it's true for both conservative and non-conservative forces.

Conservative Forces

If the work performed by a force ~~[illegible]~~ on a particle travelling from point ① to point ② depends only on:

(a) the location of ① & ②

and not on:

(b) the path from ① to ②
(c) the velocity of the particle
(d) nor on time,

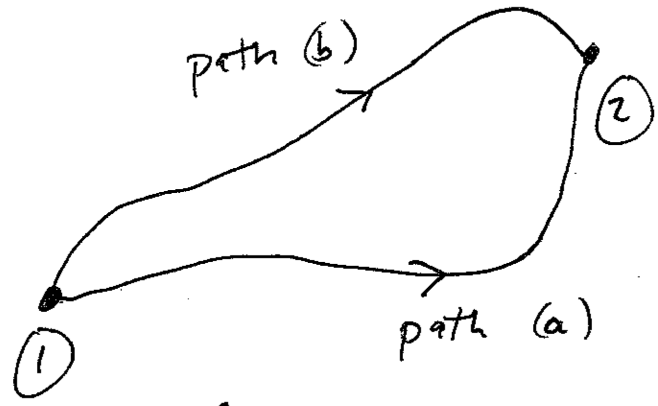
then we say the force is "conservative"

That the ~~[illegible]~~ work should be path-independent allows us to conclude that

$$\vec{\nabla} \times \vec{F} = \text{curl of } \vec{F} = \emptyset$$

via Stoke's Theorem. The proof is as follows:

Assume that the work between ① & ② is independent of path:



$$\int_{1\ \text{path(a)}}^{2} \vec{F}\cdot d\vec{r} - \int_{1\ \text{path b}}^{2} \vec{F}\cdot d\vec{r} = \emptyset$$

Then $$W_{12} = \int_{1\ \text{path a}}^{2} \vec{F}\cdot dr = \int_{1\ \text{path b}}^{2} \vec{F}\cdot d\vec{r}$$

If we reverse the direction of path (b), then $d\vec{r}$ becomes $-d\vec{r}$, but $\vec{F}$ stays the same. So

$$\int_{1\ \text{path b}}^{2} \vec{F}\cdot d\vec{r} = -\int_{2\ \text{reversed path b}}^{1} \vec{F}\cdot dr$$

Therefore the total work around path (a) and returning to 1 along reversed path (b) is zero:

$$\int_{1\ \text{path a}}^{2} \vec{F}\cdot d\vec{r} - \int_{1\ \text{path b}}^{2} \vec{F}\cdot d\vec{r} = \emptyset$$

$$\int_{1\ \text{path a}}^{2} \vec{F}\cdot d\vec{r} + \int_{2\ \text{reversed path b}}^{1} \vec{F}\cdot d\vec{r} = \emptyset$$

or $\oint \vec{F} \cdot d\vec{r} = \emptyset$

closed path

entire circuit

By Stokes Theorem: $\oint \vec{F} \cdot d\vec{r} = \int_{\text{surface}} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA$

Therefore $\int_{\text{surface}} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA = \emptyset$

Since this must hold true for any path, it must be true that

$$\boxed{\vec{\nabla} \times \vec{F} = \emptyset}$$ For conservative forces.

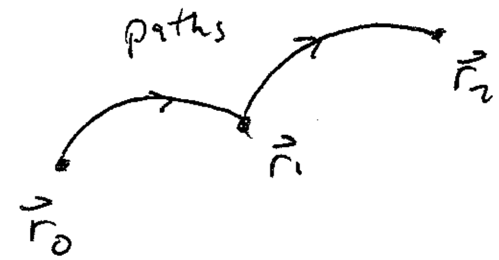
Since conservative forces perform work independent of the path (by definition), we can define a function which tells us the work performed starting from any reference point a travelling to any other point. This is the potential energy function:

For convenience we define the potential energy with a minus sign:

$$\boxed{\underset{\substack{\uparrow \\ \text{scalar}}}{U(\vec{r})} \equiv -W(\underset{\substack{\uparrow \\ \text{reference} \\ \text{point}}}{\vec{r}_0} \rightarrow \vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'}$$

Notice that if the work depended on the path, then the potential energy function would not be uniquely defined.

Travelling between any 2 points ① & ②, the change in U is unique and well defined:



$$W(\vec{r}_0 \rightarrow \vec{r}_2) = W(\vec{r}_0 \rightarrow \vec{r}_1) + W(\vec{r}_1 \rightarrow \vec{r}_2)$$

$$\therefore W(\vec{r}_1 \rightarrow \vec{r}_2) = \underbrace{W(\vec{r}_0 \rightarrow \vec{r}_2)}_{-U(\vec{r}_2)} - \underbrace{W(\vec{r}_0 \rightarrow \vec{r}_1)}_{-U(\vec{r}_1)}$$

$$= -\left[U(\vec{r}_2) - U(\vec{r}_1)\right]$$

$$= -\Delta U$$

And according to the Work-Energy Theorem,

$$\Delta T = \int_1^2 \vec{F} \cdot d\vec{r} = W(\vec{r}_1 \to \vec{r}_2) = -\Delta U$$

or

$$\Delta T + \Delta U = \emptyset$$

$$\boxed{\Delta (T + U) = \emptyset}$$

Or the mechanical energy is conserved:

$$\boxed{E \equiv \text{Mechanical energy} \equiv T + U}$$

$$\boxed{\Delta E = \emptyset}$$

Non-conservative Forces

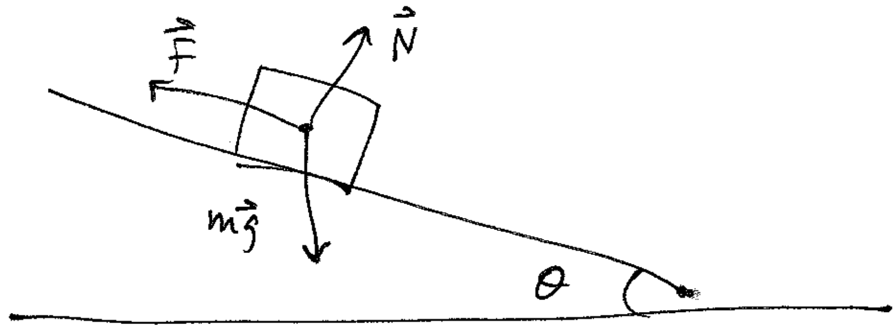
Even if we have a non-conservative force, the work-energy theorem is still useful, because the change in E must be due to the work done by the non-conservative forces.

$$\Delta T = W = W_{\text{conservative}} + W_{\text{non-conservative}}$$

$$= -\Delta U + W_{nc}$$

$$\Delta (T + U) = W_{\text{non-conservative}}.$$

Ex: Block sliding down an incline with friction



For a uniform gravitational field, $\vec{F}_{gravity} = -mg\,\hat{z}$,

so $$U_{gravity} = -\int_{z_0}^{z} (-mg)\,\hat{z}\cdot\underbrace{\hat{z}\,dz}_{d\vec{r}}$$

$$\boxed{U(z) = mg(z - z_0)}$$

$\Rightarrow$

Starting from rest and at height z (measuring from the bottom of the ramp)

$$\left.\begin{aligned} U_{initial} &= mgz \\ T_{initial} &= 0 \end{aligned}\right\} E_i = mgz$$

$$\left.\begin{aligned} U_{final} &= 0 \\ T_{final} &= \tfrac{1}{2}mv_f^2 \end{aligned}\right\} E_f = \tfrac{1}{2}mv_f^2$$

The change in mechanical energy must be due to friction:

$$\Delta E = \frac{1}{2} m v_F^2 - mgz = W_{fric}$$

The frictional force is

$$|\vec{F}| = \mu |\vec{N}| = \mu m g \cos\theta$$

It's work is

$$W_{fric} = (-\mu m g \cos\theta)(d)$$

distance travelled along the ramp.

The height (z) is related to (d) by geometry:

$$z = d \sin\theta$$

Therefore

$$\frac{1}{2} m v_F^2 - mg(d \sin\theta) = -\mu m g \cos\theta \, d$$

$$\boxed{v_F = \sqrt{2gd(\sin\theta - \mu\cos\theta)}}$$

This is the final velocity.