Rotational Dynamics - Euler's Equations

Euler's Equations are the rotational equations of motion cast into a special frame - the body frame. The body frame uses the principal axes for the coordinate system, to take advantage of the simpler relationship between $\vec{L}$ and $\vec{\omega}$ in that frame. There are, however, some sublties to using ~~in~~ the body frame (as we will see.)

We seek to find a useful expression for $\vec{\Gamma} = \frac{d\vec{L}}{dt}$, the rotational form of Newton's 2nd Law. We know that

$$\vec{L} = I\vec{\omega} \quad \text{or}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

In a fixed reference frame all of the elements of I depend on time (because the body rotates, so its coordinates change.) Also $\vec{\omega}$ depends on time also.

$$\vec{L}(t) = I(t)\,\vec{\omega}(t)$$

There are 9 independent quantities on the right hand side, all of them time-dependent. So this is rather complicated.

Now imagine that at a particular moment we choose a reference frame which is identical to the principal axes. For this one instant, the expression for $\vec{L}$ becomes simpler:

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & \emptyset & \emptyset \\ \emptyset & I_{yy} & \emptyset \\ \emptyset & \emptyset & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

or, letting λ_1, λ_2, and λ_3 be the eigenvalues of I,

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

It's still true that ω_1, ω_2, and ω_3 depend on time, but at least λ_1, λ_2, and λ_3 are constant.

To take the time derivative of $\vec{L}$, we must allow the body to rotate. Then I becomes complicated again, at least in our fixed coordinate system.

Instead, let's rotate our coordinate system with the body, so that I remains simple and diagonal and constant. This is OK, but now we must be careful about how we take the time derivative of $\vec{L}$, because our coordinate axes themselves are rotating. For example, something may appear fixed and constant with respect to our coordinates, but that quantity actually has a non-zero time derivative.

We will take the time derivative of $\vec{L}$ in 2 parts: one part will be the change in $\vec{L}$ with respect to the body frame (principal axes), and one part will be the change of the body frame with respect to a fixed coordinate system. To see how this works, let

$$\vec{L}_0 = \vec{L} \text{ at a given instant.}$$

As time goes forward, $\vec{L}_0$ will be captured (frozen) with the body frame, while $\vec{L}$ continues to evolve according to Newton's 2nd Law.

Then $\frac{d\vec{L}}{dt}$ can be written

$$\frac{d\vec{L}}{dt} = \underbrace{\frac{d(\vec{L}-\vec{L}_0)}{dt}}_{\substack{\text{change of}\\ \vec{L}\text{ w/respect}\\ \text{to body}\\ \text{frame}}} + \underbrace{\frac{d\vec{L}_0}{dt}}_{}$$

change of the body frame.

The good news is that ~~the~~ $\frac{d\vec{L}_0}{dt}$ is simple: ~~If~~ For ~~as just~~ any vector $\vec{A}_0$ frozen in the body frame, the time derivative is $\frac{d\vec{A}_0}{dt} = \vec{\omega} \times \vec{A}_0$. This follows from the same reasoning as $\vec{V} = \vec{\omega} \times \vec{r}$ or $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$.

In our case, $\frac{d\vec{L}_0}{dt} = \vec{\omega} \times \vec{L}_0$. But at this instant, $\vec{L}_0 = \vec{L}$, so we have $\boxed{\frac{d\vec{L}_0}{dt} = \vec{\omega} \times \vec{L}}$.

The first term, $\frac{d(\vec{L}-\vec{L}_0)}{dt}$ is the time rate change of $\vec{L}$ relative to the body frame.

Taylor uses the "dot" notation to indicate these special time derivatives: $\frac{d(\vec{L}-\vec{L}_0)}{dt} = \dot{\vec{L}}$

I prefer to use $\frac{d(\vec{L}-\vec{L}_0)}{dt} = \frac{\delta \vec{L}}{\delta t}$, ~~to indicate~~

So $\boxed{\frac{d\vec{L}}{dt} = \frac{\delta \vec{L}}{\delta t} + \vec{\omega} \times \vec{L}}$ * in this notation.

This is a vector statement, so it is true no matter what coordinate axes we use. Essentially it just says that "velocities" add (in this case, the "velocity" of $\vec{L}$). It is equivalent to

$$\vec{v} = \vec{V}_{cm} + \vec{v}'$$

However now we would like to project the vector ~~statement~~ equation onto the instantaneous body axes.

$$\frac{\delta \vec{L}}{\delta t} = \text{time rate change of } \vec{L} \text{ w/respect to body frame} = \frac{d}{dt}(\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$$= (\lambda_1\dot{\omega}_1, \lambda_2\dot{\omega}_2, \lambda_3\dot{\omega}_3)$$

$$\vec{\omega} \times \vec{L} = (\omega_1, \omega_2, \omega_3) \times (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$$= \left((\lambda_3-\lambda_2)\omega_2\omega_3, (\lambda_1-\lambda_3)\omega_1\omega_3, (\lambda_2-\lambda_1)\omega_2\omega_1\right)$$

On the left hand side we have

$$\left(\left(\frac{d\vec{L}}{dt}\right)_1, \left(\frac{d\vec{L}}{dt}\right)_2, \left(\frac{d\vec{L}}{dt}\right)_3 \right)$$

To be clear: 1, 2, 3 refer to the body axes.

• The notation means that we take the time derivative first, which give us <u>the</u> <u>true</u> $\frac{d\vec{L}}{dt}$, and then we project the resulting vector $\frac{d\vec{L}}{dt}$ onto the three body axes, at each moment.

So we have

$$\left(\frac{d\vec{L}}{dt}\right)_1 = \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2$$

$$\left(\frac{d\vec{L}}{dt}\right)_2 = \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3$$

$$\left(\frac{d\vec{L}}{dt}\right)_3 = \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1$$

Now we can equate $\frac{d\vec{L}}{dt}$ with $\vec{\Gamma}$, the torque

In particular, we cast the torque onto the body frame: $\vec{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$:

$$\left.\boxed{\begin{aligned} \Gamma_1 &= \lambda_1 \dot{\omega}_1 + (\lambda_3 - \lambda_2)\,\omega_3 \omega_2 \\ \Gamma_2 &= \lambda_2 \dot{\omega}_2 + (\lambda_1 - \lambda_3)\,\omega_1 \omega_3 \\ \Gamma_3 &= \lambda_3 \dot{\omega}_3 + (\lambda_2 - \lambda_1)\,\omega_2 \omega_1 \end{aligned}}\right\} \text{"Euler Equations"}$$

These equations tell us how the components of the torque projected onto the body frame govern the time development of the $\vec{\omega}$ vector, when $\vec{\omega}$ is also projected onto the body frame.

- Note that $\vec{\omega}$ exists in the fixed frame, not the body frame. For example, an observer fixed in the body frame would not observe any rotation at all (although he/she may experience pseudo-forces, the topic of Taylor's Chapter 9.) So $\vec{\omega}$ is not observed in the body frame, instead it is observed in the fixed frame and projected onto the body frame at each moment.
- Similarly, $\vec{\Gamma}$ exists in the fixed frame.

- Note that if we solve for ω_1, ω_2, and ω_3, then we know how $\vec{\omega}$ evolves in time as projected onto the moving body frame. To determine how $\vec{\omega}$ evolves in time in the fixed frame, we have additional work to do.

Zero Torque Case - Tennis Racket Theorem.

Suppose that λ_1, λ_2, and λ_3 are all unique, and that at $t=0$ $\vec{\omega} = \omega_3 \hat{e}_3$ (it points only along the 3rd principal axis.). Then $\omega_1 = \omega_2 = 0$, and Euler's equations say (with zero torque)

$$\lambda_1 \dot{\omega}_1 = 0 \Rightarrow \omega_1 = \text{constant (zero)}$$
$$\lambda_2 \dot{\omega}_2 = 0 \Rightarrow \omega_2 = \text{constant (zero)}$$
$$\lambda_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant}$$

$\Rightarrow$ In this case $\vec{\omega}$ points along $\hat{e}_3$ forever.

$\Rightarrow$ If a body in a torque-free situation starts rotating about a principal axis, then it will do so forever, with constant angular velocity.

Now we can ask: Is the motion around a principal axis stable? In other words, will a small perturbation remain small, or will the body's rotation axis tend to wobble with a large angle?

Suppose that at $t=\emptyset$, $\vec{\omega}$ is not along a principal axis. Then at least 2 components of $\vec{\omega}$ are non-zero, which means that at least one component must be changing in time w/respect to the body axis $\Rightarrow$ This follows from Euler's Eq:

$$\lambda_1 \dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_3\omega_2 = \emptyset \qquad \text{(for example.)}$$

(with $\omega_2 \neq \emptyset$ and $\omega_3 \neq \emptyset$, $\dot{\omega}_1$ is non-zero.).

Now suppose $\vec{\omega} = \omega \hat{e}_3$, and at $t > \emptyset$ we give it a small kick that makes ω_1 and ω_2 small and non zero. Will ω_1 and ω_2 grow, or do they oscillate about zero?

From the 3rd Euler Equation, if ω_1 and ω_2 are small, then $\dot{\omega}_3$ remains <u>very</u> small:

$$\lambda_3 \dot{\omega}_3 + (\lambda_2 - \lambda_1)\underset{\substack{\uparrow \\ \text{small}}}{\omega_2}\underset{\substack{\uparrow \\ \text{small}}}{\omega_1} = \emptyset$$

so lets take ω_3 approximately constant
The The 1st 2 Euler Equations say

$$\lambda_1 \dot{\omega}_1 = \left[(\lambda_2-\lambda_3)\omega_3\right]\omega_2$$

$$\lambda_2 \dot{\omega}_2 = \underbrace{\left[(\lambda_3-\lambda_1)\omega_3\right]}\omega_1$$

square bracket is approximately constant.

Now combine equations by differentiating the 1st equation:

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)\ \omega_3^2}{\lambda_1\lambda_2}\right]\omega_1$$

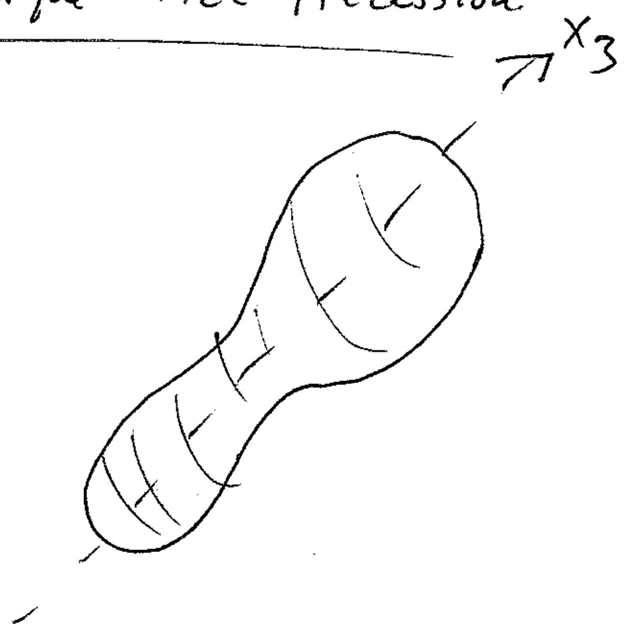
If the coefficient in square brackets is (+), then ω_1 will oscillate about zero (and similarly for ω_2).

Note that the bracket is (+) if λ_3 is greater than both λ_1 & λ_2 or λ_3 is less than both λ_2 and λ_1. Therefore spinning about the ~~largest moment~~ principal axis with the largest moment is stable and also the axis with the smallest moment is stable.
But the intermediate-moment axis is unstable.

This is the tennis-racket theorem.

Two Equal moments, no torque: Free Precession

"Free symmetric top"



By symmetry, $I_1 = I_2$.

Define $I_1 \equiv I$, then

a principal moment

$$I = \begin{pmatrix} I & \emptyset & \emptyset \\ \emptyset & I & \emptyset \\ \emptyset & \emptyset & I_3 \end{pmatrix}$$

inertia tensor

Euler's Equations with $\Gamma_1 = \Gamma_2 = \Gamma_3 = \emptyset$:

$$I_3 \dot{\omega}_3 = (I - I)\omega_1\omega_2 = \emptyset$$

$$\boxed{\Rightarrow \omega_3 = \text{constant}}$$

Also $\dot{\omega}_1 = \left[\frac{(I - I_3)\omega_3}{I}\right]\omega_2$ $\Leftarrow$ Define $\Omega \equiv \left(\frac{I - I_3}{I}\right)\omega_3$

and $\dot{\omega}_2 = -\left[\frac{(I - I_3)\omega_3}{I}\right]\omega_1$

Then
$$\left.\begin{aligned} \dot{\omega}_1 &= \Omega\omega_2 \\ \dot{\omega}_2 &= -\Omega\omega_1 \end{aligned}\right\} \text{Coupled Differential Eqs.}$$

We solved in before for the charged particle in a Constant $\vec{B}$ Field (Equations had the same form.)

Solution: $\vec{\omega} = (\omega_0 \cos(\Omega t), -\omega_0 \sin(\Omega t), \omega_3)$

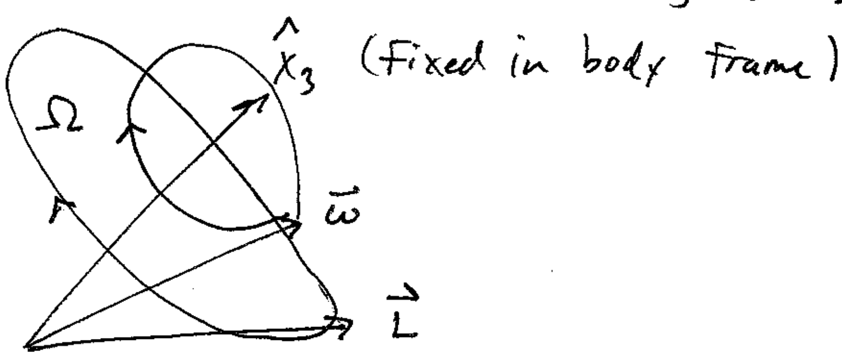
where $\omega_0 = \omega_1$ at $t = \phi$ and we have chosen the directions of $\hat{x}_1$ & $\hat{x}_2$ so that $\hat{x}_1$ points along the transverse component of $\vec{\omega}$ at $t = \phi$.

Therefore, as seen from the body frame, $\vec{\omega}$ precesses around $\hat{x}_3$, tracing out a cone called "the body cone."

$\vec{L} = (I\omega_1, I\omega_2, I_3\omega_3)$

$= (I\omega_0 \cos(\Omega t), -I\omega_0 \sin(\Omega t), I_3\omega_3)$

Therefore $\vec{\omega}$, $\vec{L}$, and $\hat{x}_3$ all lie in a plane, and as viewed from the body frame, $\vec{L}$ also traces out a cone around $\hat{x}_3$. ~~called the~~



In ~~space~~ the space frame, a fixed, inertial coordinate system, $\vec{L}$ is constant (because there is no torque), and $\hat{x}_3$ and $\vec{\omega}$ precess about it.

View from the fixed inertial frame.

Here we will ignore the Euler equations and solve ~~the~~ for the motion from scratch. (Because the Euler equations apply to the body frame only.)

The principal axes, which change in time, are called $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$. We can project $\vec{\omega}$ onto them: (And also project $\vec{L}$):

$$\vec{\omega} = (\omega_1 \hat{x}_1 + \omega_2 \hat{x}_2) + \omega_3 \hat{x}_3$$
$$\vec{L} = I(\omega_1 \hat{x}_1 + \omega_2 \hat{x}_2) + I_3 \hat{x}_3$$

Eliminate $\omega_1 \hat{x}_1 + \omega_2 \hat{x}_2$ in terms of $\Omega = \left(\frac{I - I_3}{I}\right)\omega_3$

$$\vec{\omega} - \frac{\vec{L}}{I} = \left(\omega_3 - \frac{I_3}{I}\right)\hat{x}_3 = \left(\frac{I - I_3}{I}\right)\omega_3 \hat{x}_3$$

$$= \Omega \hat{x}_3$$

$$\Longrightarrow \boxed{\begin{aligned} \vec{\omega} &= \frac{\vec{L}}{I} + \Omega \hat{x}_3 \\ \vec{L} &= I(\vec{\omega} - \Omega \hat{x}_3) \end{aligned}} = \frac{L}{I}\hat{L} + \Omega \hat{x}_3 \quad \text{where } \hat{L} \equiv \frac{\vec{L}}{|\vec{L}|}$$

Again, we find that $\vec{\omega}$, $\vec{L}$, and $\hat{x}_3$ lie in a plane, so any motion of $\vec{\omega}$ & $\hat{x}_3$ around $\vec{L}$ must be something like a precession.

What is the rate of precession? The time rate change of $\hat{x}_3$ is

$$\frac{d\hat{x}_3}{dt} = \vec{\omega} \times \hat{x}_3 \quad \text{, because } \hat{x}_3 \text{ is fixed}$$

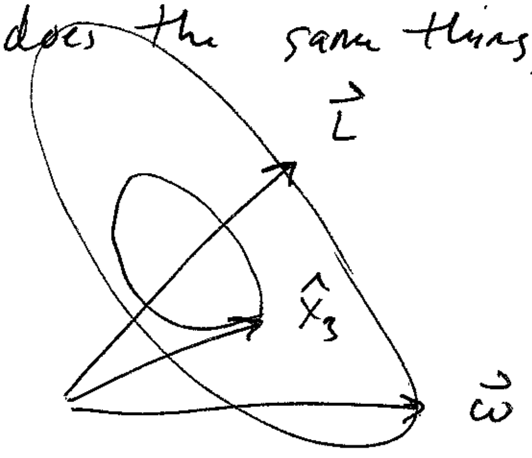
in the body frame. So

$$\frac{d\hat{x}_3}{dt} = \left(\frac{L}{I}\hat{L} + \Omega\hat{x}_3\right) \times \hat{x}_3 = \underbrace{\frac{L}{I}}\hat{L} \times \hat{x}_3$$

here $\frac{L}{I}\hat{L}$ plays the role of $\vec{\omega}$.

So let $\vec{\omega}' \equiv \frac{L}{I}\hat{L}$. The frequency of rotation is $|\vec{\omega}'| = \frac{L}{I}$, so <u>$\hat{x}_3$ precesses around the fixed $\vec{L}$ vector with frequency $\frac{L}{I}$ in the fixed frame.</u>

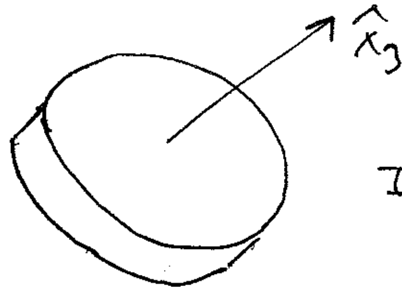
$\vec{\omega}$ does the same thing, because it is co-planar.



<u>view from fixed frame.</u>

Note that we have 2 cases:

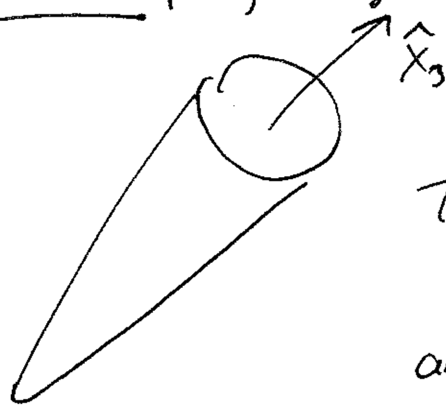
- Oblate top, $I_3 > I$ like a coin,



$I_3 > I$.

Then $\Omega = \left(\frac{I - I_3}{I}\right)\omega_3 < \emptyset$, so the precession is clockwise.

- Prolate top, $I_3 < I$, like a carrot



Then $\Omega = \left(\frac{I - I_3}{I}\right)\omega_3 > \emptyset$, and the precession is counter-clockwise.

Chandler Wobble: The earth is a free symmetric top with $I_3 > I$ such that $\frac{I - I_3}{I} \approx -\frac{1}{320}$

(The earth has a small bulge near the equator.)

So $\Omega = -\frac{1}{320}\,\omega_{earth} = \frac{-1}{320}\,\frac{2\pi}{(1\,day)}$

or $\frac{\Omega}{\omega_{earth}} = -\frac{1}{320}$

So the earth's $\vec{\omega}$ vector should precess about the geometric north pole ($\hat{x}_3$) once every 320 days. In practice, the true period is about 430 days, the difference being ascribed to the fact that the earth is not perfectly rigid.

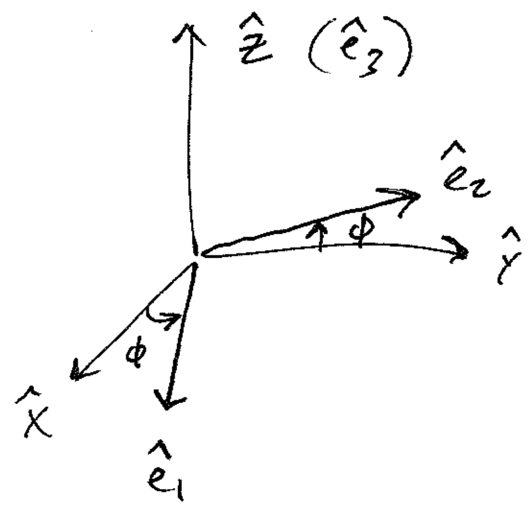
How big is the $\vec{\omega}$ cone for the earth? Answer: the distance between the north pole and the spot where $\vec{\omega}$ penetrates earth surface is about 10 meters. So the half angle of the cone is only 10^{-4} degrees. This is difficult to observe, but not impossible (it can be seen by locating the point about which the stars revolve each night.) It was first observed in 1891, after having been predicted by Newton and Euler.

Euler Angles and a spinning top with pivot & gravity

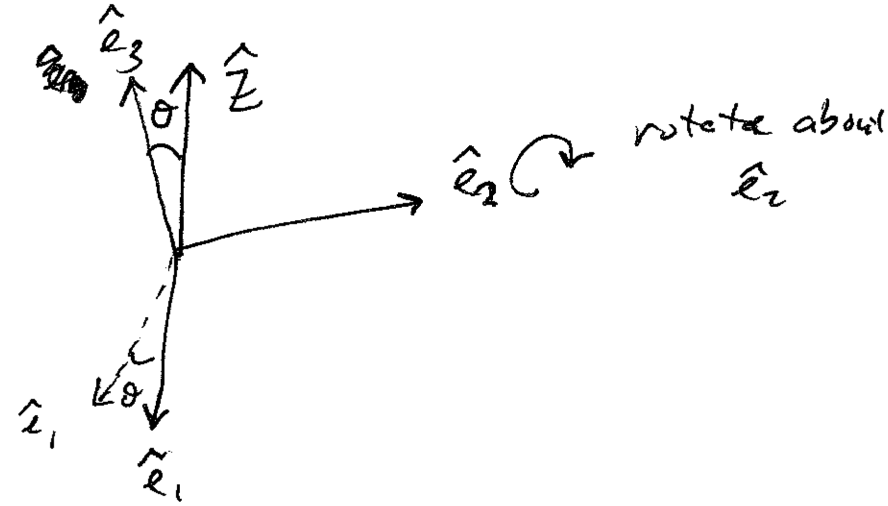
We can relate the absolute orientation of the body axes to the space axes (fixed) via the "Euler angles". (There are several convention for how to define the Euler angles, this one is used by Taylor.)

Let $\hat{e}_1$, $\hat{e}_2$, & $\hat{e}_3$ be the principal axes (body axes), and $\hat{x}$, $\hat{y}$, $\hat{z}$ be the space (fixed) axes. We start with both coordinate systems aligned, and we wish to rotate the body frame to an arbitrary orientation.

1) First rotate about the $\hat{z}$ axis (equivalent to $\hat{e}_3$) by angle ϕ:

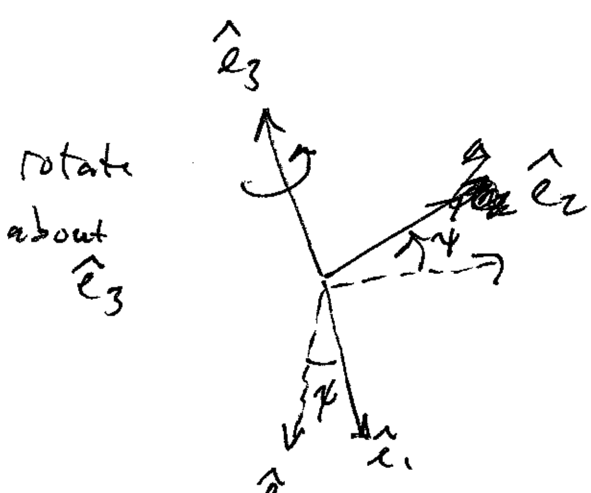


2) Now tip the $\hat{e}_3$ axis down away from $\hat{z}$ by ~~ϕ~~ polar angle θ, rotating about $\hat{e}_2$:



After step 2, $\hat{e}_3$ has been placed in its final orientation.

3) Now rotate about $\hat{e}_3$ by angle ψ:



This put $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ in their final orientation.

Our goal is to write the Lagrangian for a spinning top using ϕ, θ, ψ as the generalized coordinates. First we will need an expression for the $\vec{\omega}$ vector in terms of ϕ, θ, ψ.
We can do this by adding the velocities due to each rotation one after another, because $\vec{\omega}$ is a vector (so it adds).

- Step 1 velocity: $\vec{\omega}_a = \dot{\phi}\hat{z}$
- Step 2 velocity: $\vec{\omega}_b = \dot{\theta}\hat{e}_2'$

the location of $\hat{e}_2$ after step 1.

- Step 3 velocity: $\vec{\omega}_c = \dot{\psi}\hat{e}_3$

Total angular velocity: $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$

To find $\vec{L}$ or KE, it is simplest to work in the body frame. However, if we take the case of a symmetric top $(I_1 = I_2)$, then things are particularly simple because the final rotation (ψ) has no effect on the ~~tntor~~ inertia tensor.

~~Then $\hat{e}_2'$ and $\hat{e}_3$~~

Then $\hat{e}_1'$ and $\hat{e}_2'$ are body axes (principal axes)

position of $\hat{e}_1$ & $\hat{e}_2$ after first 2 rotations.

Then $\hat{z} = \cos(\theta)\hat{e}_3 - \sin(\theta)\hat{e}_1'$

so $\vec{\omega}_a = \dot{\phi}(\cos(\theta)\hat{e}_3 - \sin(\theta)\hat{e}_1')$

or

$$\boxed{\vec{\omega} = (-\dot{\phi}\sin\theta)\hat{e}_1' + \dot{\theta}\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3}$$

location of $\hat{e}_1$ after first 2 rotations

location of $\hat{e}_2$ after first rotation (2nd rotation does nothing to $\hat{e}_2$)

Then with principal moments λ_1, λ_1, and λ_3

same

we have

$$\boxed{\vec{L} = (-\lambda_1\dot{\phi}\sin\theta)\hat{e}_1' + \lambda_1\dot{\theta}\hat{e}_2' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3}$$

Note that $L_3 = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)$

and $L_z = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$

(homework)

$$= \lambda_1\dot{\phi}\sin^2\theta + L_3\cos\theta$$

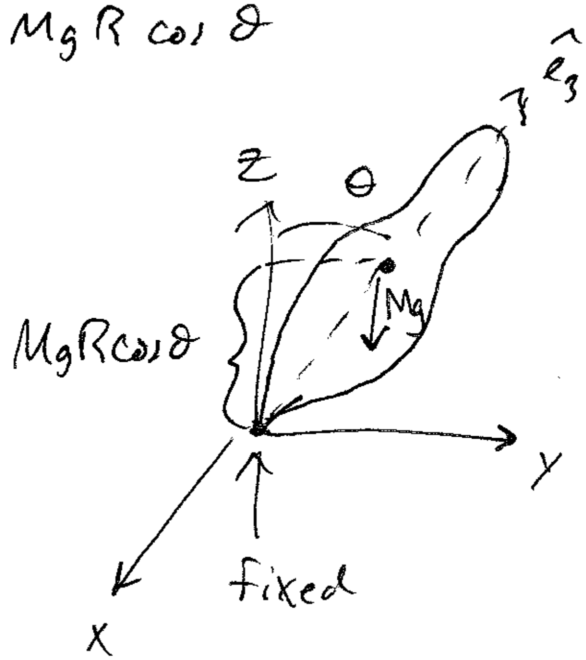
or $$\dot{\phi} = \frac{L_z - L_3\cos\theta}{\lambda_1\sin^2\theta}$$

Since $KE = T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$

and $\lambda_1 = \lambda_2$ (by assumption)

Then $T = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\psi + \dot\phi\cos\theta)^2$

The PE for the symmetric spinning top, (from gravity) is $U = MgR\cos\theta$



So the Lagrangian is

$$\boxed{\mathcal{L} = \frac{1}{2}\lambda_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}\lambda_3(\dot\psi + \dot\phi\cos\theta)^2 - MgR\cos\theta}$$

There are 3 Lagrangian equations of motion:

<u>θ Equation:</u> $$\boxed{\lambda_1\ddot\theta = -\lambda_3(\dot\psi + \dot\phi\cos\theta)(\sin\theta)\dot\phi + MgR\sin\theta + \lambda_1\dot\phi^2\sin\theta\cos\theta}$$

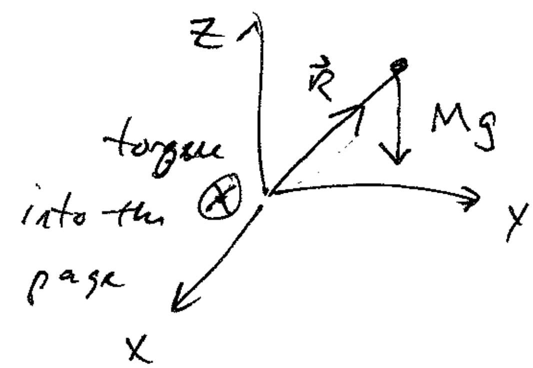
Both ϕ & ψ are <u>ignorable</u> (do not appear in $\mathcal{L}$),

so p_ϕ & p_ψ are constant:

<u>ϕ Equation:</u>

$$p_\phi = \underbrace{\lambda_1 \dot{\phi} \sin^2\theta + \lambda_3 (\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta}_{L_z} = \text{constant}$$

This says L_z = constant, which is ~~the~~ true ~~becau~~ because all the torque ~~is~~ vector is in the xy plane ;



<u>ψ Equation:</u>

$$p_\psi = \underbrace{\lambda_3 (\dot{\psi} + \dot{\phi}\cos\theta)} = \text{constant}$$

this is L_3,
the component of
$\vec{L}$ along $\hat{e}_3$

This is constant because there $\vec{R}$ is parallel to $\hat{e}_3$, so $\vec{R} \times M\vec{g}$ has no component along $\hat{e}_3$.

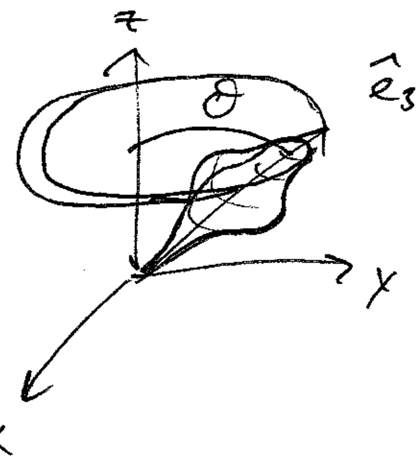
Since $L_3 = \lambda_3 \omega_3$, and since L_3 = constant, we also have ω_3 is constant, where

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

Precession

Let's see if the top can precess about the z axis with $\hat{e}_3$ making a constant angle θ with the z axis.

$\dot{\theta} = 0$ (by assumption).



Also

$$\dot{\phi} = \frac{\overset{\text{constant}}{L_z} - \overset{\text{constant}}{L_3}\cos\theta}{\lambda_1 \sin^2\theta}$$

θ constant by assumption,

so $\boxed{\dot{\phi} = \text{constant} \equiv \Omega}$

precession frequency

Then the ψ Equation says that $\dot{\psi}$ is also constant:

$$\cancel{\lambda_3} \quad \lambda_3\left(\dot{\psi} + \overset{\text{constant}}{\dot{\phi}}\cos\theta\right) = \text{constant}$$

constant

so $\boxed{\dot{\psi} = \text{constant}}$

So for this motion, the rate of rotation about the symmetry axis is constant, And the symmetry axis precesses about $\hat{z}$ at a constant rate Ω tracing a perfect cone.

And what is Ω? Well the θ equation is

$$\underbrace{\lambda\ddot{\theta}}_{\substack{\text{zero} \\ \text{(by assumption)}}} = -\lambda_3 \underbrace{(\dot{\psi} + \dot{\phi}\cos\theta)}_{\omega_3} \sin\theta \underbrace{\dot{\phi}}_{\Omega} + MgR\sin\theta + \lambda_1 \Omega^2 \sin\theta\cos\theta$$

or $\boxed{\lambda_1 \Omega^2 \cos\theta - \lambda_3 \omega_3 \Omega + MgR = 0}$ A quadratic Equation for Ω.

We have 2 precession rates which are possible, as long as Ω is real.

For the typical case where ω_3 is large, (rapidly spinning on the axis),

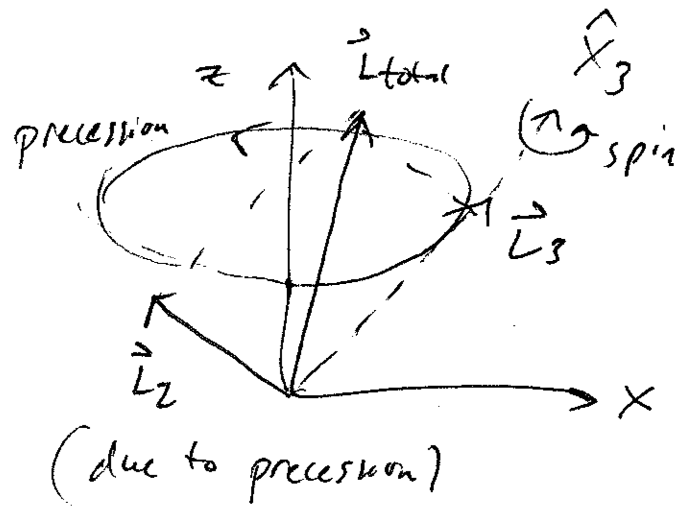
$$\boxed{\Omega \approx \frac{MgR}{\lambda_3 \omega_3}}$$ (slow precession)

The Faster root is

$$\boxed{\Omega \approx \frac{\lambda_3 \omega_3}{\lambda_1 \cos\theta}}$$ (Fast precession) ← does not depend on (g).

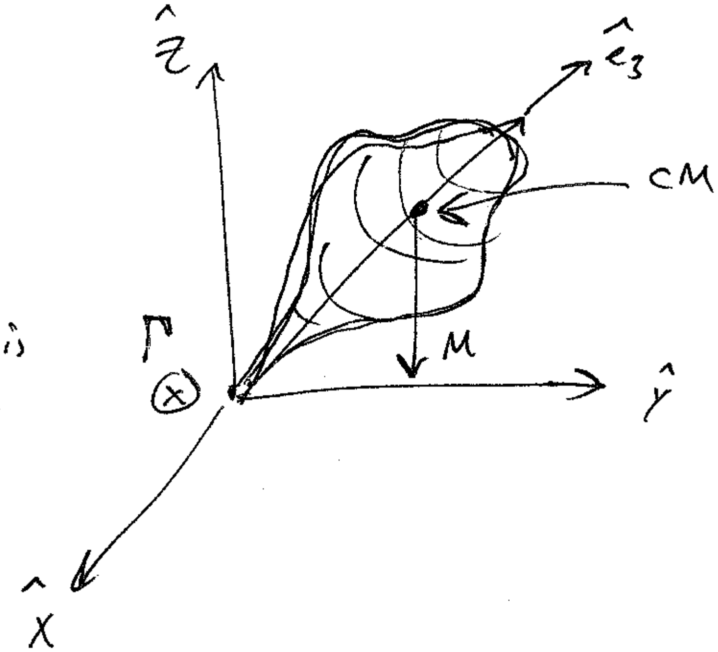
The 2^{nd} one is the free precession of a body which does not experience any torques.

What's going on here? Well, we have 2 types of $\vec{L}$, that along $\hat{e}_3$ due to the spinning, and another due to the precession.



If Ω is large enough, then the x components of L_2 & L_3 cancel, and then $\vec{L}$ is almost entirely vertical. In this case there is no torque, so we have free precession of $\hat{e}_3$ about $\vec{L}$

The slow precession is the more obvious one which is driven by the gravitational torque. We can analyze it from scratch as follows:



torque is into the page.

If $\vec{\omega}$ is large, then $\vec{L} \approx \lambda_3 \omega \hat{e}_3$

The torque is $\vec{\Gamma} = \vec{R} \times M\vec{g} = \frac{d\vec{L}}{dt}$

$\vec{L}$ will change, so $\vec{L}$ will develop a small component along $\hat{e}_1$ and/or $\hat{e}_2$. But if ω_3 is very large, the component of $\vec{L}$ along $\hat{e}_1$ & $\hat{e}_2$ will ~~remain small~~ be approximately zero in comparison. So

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\lambda_3 \omega \vec{e}_3\right) = \lambda_3 \omega \frac{d\vec{e}_3}{dt} = \vec{\Gamma} = \vec{R} \times M\vec{g}$$

by Newton's 2nd Law

$g = -g\hat{z}$, so $\frac{d\hat{e}_3}{dt} = \frac{MgR}{\lambda_3 \omega} \hat{z} \times \hat{e}_3$

So This is like $\frac{d\hat{e}_3}{dt} = \vec{\omega} \times \hat{e}_3$ with $\vec{\omega} = \frac{MgR}{\lambda_3 \omega} \hat{z}$

So $\hat{e}_3$ precesses about $\hat{z}$ with frequency $\frac{MgR}{\lambda_3 \omega}$, which is our slow precession frequency from the Lagrangian analysis.

Nutation

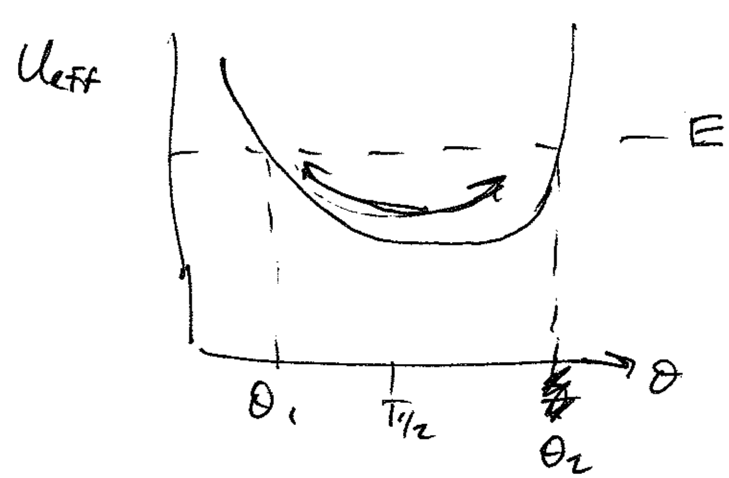
Now we allow θ to vary a little bit about the value which gives uniform precession. For small displacements, θ will oscillate about the stable value. This is called nutation.

It can be shown (homework) that the energy of the top is

$$E = \tfrac{1}{2}\lambda_1 \dot{\theta}^2 + U_{eff}(\theta)$$

with
$$U_{eff}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

The effective potential is

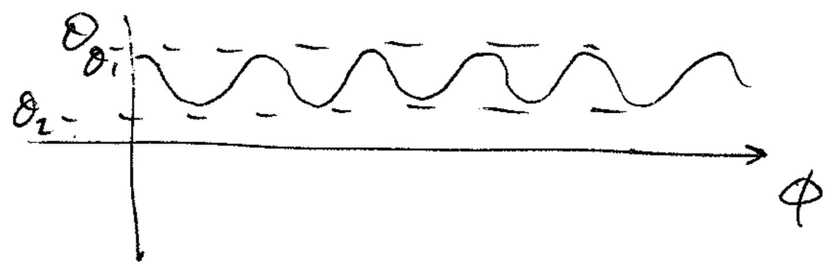


So θ oscillates between two extreme values. How this looks depends on how fast ϕ advances:

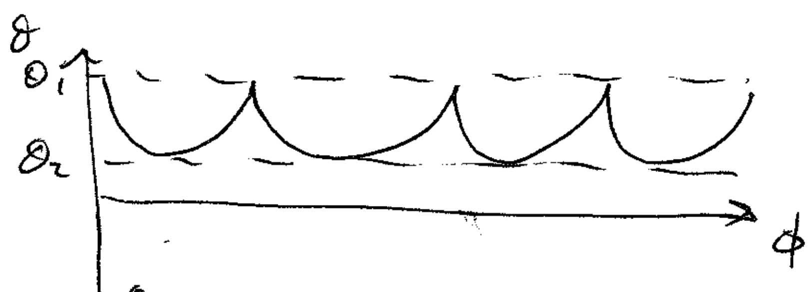
$$\dot{\phi} = \frac{L_z - L_3 \cos\theta}{\lambda_1 \sin^2\theta}$$

So if $L_z > |L_3|$, then $L_3 - L_3 \cos\theta > 0$

and $\dot{\phi}$ always advances forward. The the motion looks like



But if $\dot{\phi}$ can become zero, we have



or if $\dot{\phi}$ can become negative