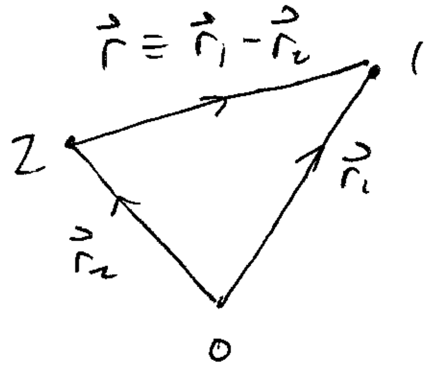
Two-Body Central Force Problems

"The Kepler Problem" - Historically, one of the most important systems in Classical Mechanics, because it established the Newtonian Framework. The lack of friction in the system makes it particularly well-suited to our analysis methods, including the Lagrangian approach.

Two objects, mass $\underline{m_1}$ and $\underline{m_2}$.



Assume the only force is a mutual attraction, and is conservative. Then there is a potential energy: $U(\vec{r}_1, \vec{r}_2)$. In fact, U only depends on the magnitude of $\vec{r}_1 - \vec{r}_2$. So define $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and $r \equiv |\vec{r}|$. Then

$U = U(r)$. Then the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$$

We can simplify greatly by defining

$$\vec{R} = \vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$$

where $M \equiv m_1 + m_2$. Then

$\vec{P} = M\dot{\vec{R}}$. We can write

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \quad \text{and} \quad r_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

Then the KE is

$$\begin{aligned} T &= \tfrac{1}{2}\left(m_1\dot{\vec{r}}_1^{\,2} + m_2\dot{\vec{r}}_2^{\,2}\right) \\ &= \tfrac{1}{2}\left(m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2\right) \\ &= \tfrac{1}{2}\left(M\dot{\vec{R}}^2 + \frac{m_1 m_2}{M}\dot{\vec{r}}^2\right) \end{aligned}$$

Now define

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} = \text{"reduced mass"}$$

= units of kg.

Then $T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$

Then

$$\mathcal{L} = T - U = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{cm}} + \underbrace{\left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{\equiv \mathcal{L}_{rel}}$$

The physics is described by the motion of the center of mass and the relative motion of the reduced mass.

Fortunately, $\vec{R}$ is ignorable, it does not appear in the equations of motion. In fact,

$$\mathcal{L}_{cm} = \tfrac{1}{2} M \dot{\vec{R}}^2$$

is the Lagrangian for a single free particle of mass M.

For $\vec{r}$, we have $\mathcal{L}_{rel}$, and the $\vec{r}$ equation of motion is

$$\mu \ddot{\vec{r}} = -\nabla U(\vec{r}),$$

Newton's 2^{nd} Law for a ~~pa~~ ficticious particle of mass μ.

Now if we choose the reference frame to be fixed w/respect to the CM, then $\dot{\vec{R}} = \emptyset$,

so

$$\mathcal{L} = \mathcal{L}_{rel} = \tfrac{1}{2}\mu \dot{\vec{r}}^2 - U(\vec{r}).$$

Angular Momentum:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

In the CM frame, with $\vec{R} = \emptyset$, then

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}, \quad \vec{r}_2 = \frac{m_1}{M}\vec{r}, \text{ and}$$

$$\vec{L} = \frac{m_1 m_2}{M^2}\left(m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}}\right)$$
$$= \vec{r} \times \mu \dot{\vec{r}}$$

Since $\vec{L}$ is constant, the direction of $\vec{r} \times \dot{\vec{r}}$ is constant, so they stay in the same plane $\Rightarrow$ the entire motion is in a fixed plane.

Polar Coordinates

$$\mathcal{L}_{rel} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Then $\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{constant} = \ell$ (ϕ equation)

or $$\boxed{\dot{\phi} = \frac{\ell}{\mu r^2}}$$

And $\dfrac{\partial \mathcal{L}}{\partial r} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{r}}$ or

$$\boxed{\mu r \dot{\phi}^2 - \frac{dU}{dr} = \mu \ddot{r}}$$ (r equation)

This is the radial component of $\vec{F} = m\vec{a}$.
Substituting for $\dot{\phi}$,

$$\mu \ddot{r} = -\frac{dU}{dr} + \underbrace{\mu r \dot{\phi}^2}_{\equiv F_{CF}} = -\frac{dU}{dr} + F_{CF}$$

$\equiv F_{CF}$ = "fictious centrifugal force"

$$F_{CF} \equiv \mu r \dot{\phi}^2 = \mu \frac{(r\dot{\phi})^2}{r} = \mu \frac{v_r^2}{r}$$

The best part is that the "centrifugal" force can be written as the derivative of a fictitious potential energy:

$$F_{CF} = \mu r \dot{\phi}^2 = \frac{\ell^2}{\mu r^3} \quad \left(\text{using } \dot{\phi} = \frac{\ell}{\mu r^2}\right)$$

or $$F_{CF} = -\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right) \equiv -\frac{d}{dr} U_{CF},$$

where $U_{CF} = \dfrac{\ell^2}{2\mu r^2}$. Then the radial equation is

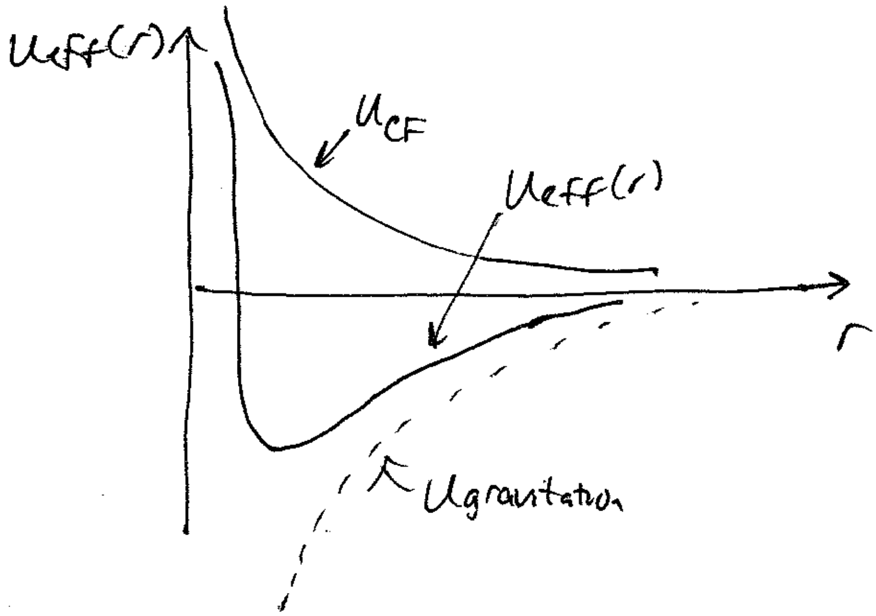
$$\mu \ddot{r} = -\frac{d}{dr}\underbrace{\left[U(r) + U_{CF}(r)\right]}_{\equiv U_{eff}(r)}$$

where $U_{eff}(r) = U(r) + U_{CF}(r) = U(r) + \dfrac{\ell^2}{2\mu r^2}$

U_{eff} is the effective potential energy for the radial motion, ignoring the ϕ motion. In other words, the radial motion is identical to what it would be if the particle were moving in a potential with the form of U_{eff}.

For a gravitational field,

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$



Energy Conservation

Multiply the r equation by $\dot{r}$:

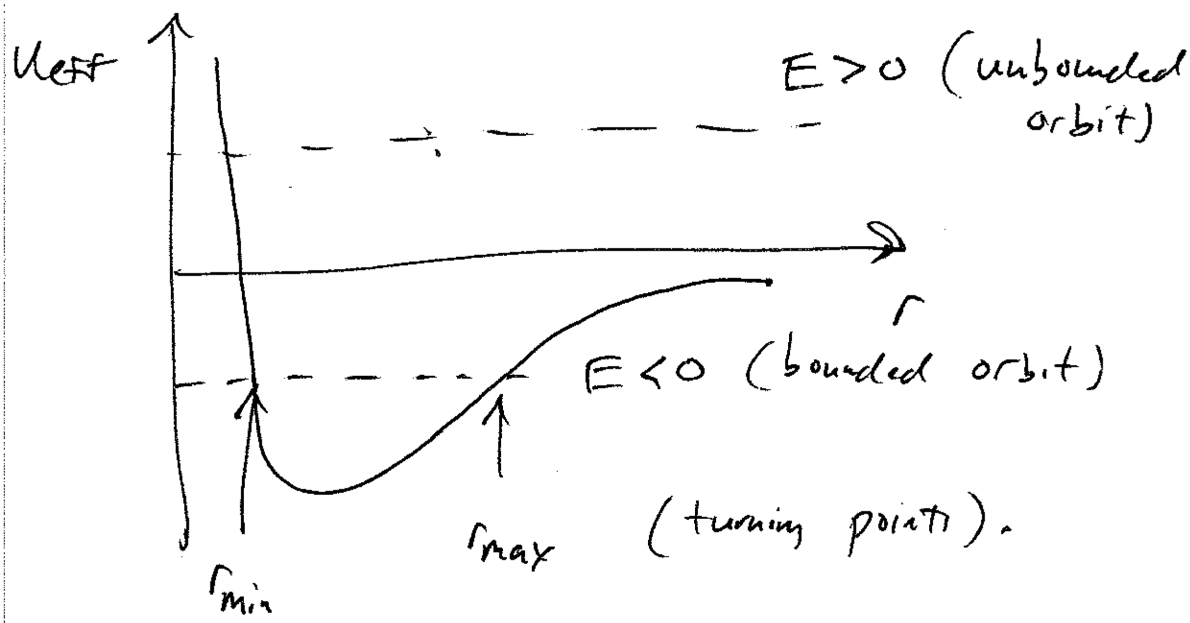
$$\underbrace{\mu\dot{r}\ddot{r}}_{\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right)} = -\frac{dU_{eff}}{dr}\dot{r} = -\frac{dU_{eff}}{dr}\frac{dr}{dt} = -\frac{dU_{eff}}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) + \frac{d}{dt}\left(U_{eff}\right) = \emptyset$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + U_{eff}\right) = \emptyset$$

or $\frac{1}{2}\mu\dot{r}^2 + U_{eff} = \text{constant} = E$ (Energy conservation).

Bounded or Unbounded orbit?



Orbital Equation

We would like to know the shape of the orbit, $r(\phi)$. So we re-write the equation of motion:

$$\mu \ddot{r} = F(r) + \frac{\ell^2}{\mu r^3} \quad , \quad F(r) = -\frac{dU}{dr}$$

Substitute $u = \frac{1}{r}$ or $r = \frac{1}{u}$,

and replace $\frac{d}{dt}$ by

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

Then

$$\dot{r} = \frac{d}{dt} r = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

and

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(-\frac{\ell}{\mu}\frac{du}{d\phi}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2}$$

Then the Equation of Motion is

$$\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$$

$$\Downarrow$$

$$-\frac{\ell^2 u^2}{\mu}\frac{d^2u}{d\phi^2} = F + \frac{\ell^2 u^3}{\mu}$$

$$\boxed{u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u^2(\phi)} F}$$

This differential Equation determines $u(\phi)$. Then $r(\phi)$ is just $\frac{1}{u(\phi)}$.

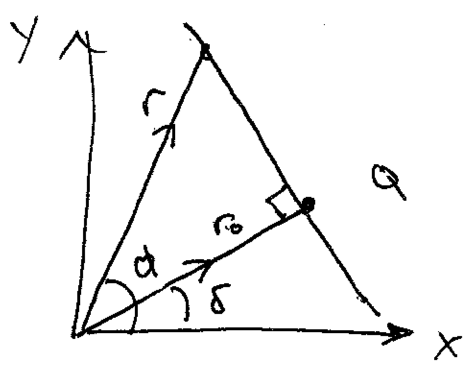
<u>Example: Free Particle</u>

Then $F = \emptyset$, so $u''(\phi) = -u(\phi)$.

Solution: $u(\phi) = A\cos(\phi - \delta)$

Then $r(\phi) = \frac{1}{u(\phi)} = \frac{r_0}{\cos(\phi-\delta)}$ where $r_0 = \frac{1}{A}$

This is a line in polar coordinates



~~The~~ The line is defined by the fact that

$$r(\phi)\cos(\phi - \delta) = \text{constant} = r_0.$$

More important is the case of the inverse-square law:

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2, \quad \gamma = GMm \text{ for gravity}$$

Then $$\boxed{u''(\phi) = -u(\phi) + \frac{\gamma\mu}{\ell^2}}$$ Eq. of Motion for inverse square law.

substitute $w(\phi) = u(\phi) - \frac{\gamma\mu}{\ell^2}$, then

$$w''(\phi) = -w(\phi)$$

Solution: $w(\phi) = A\cos(\phi - \delta)$. Make $\delta = \emptyset$ by choosing an appropriate direction for $\phi = 0$.

Then
$$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi = \frac{\gamma\mu}{\ell^2}\left(1 + \frac{A\ell^2}{\gamma\mu}\cos\phi\right)$$

Define $\varepsilon \equiv \frac{A\ell^2}{\gamma\mu}$, Then

$$u(\phi) = \frac{\gamma\mu}{\ell^2}(1 + \varepsilon\cos\phi)$$

Also define $c = \frac{\ell^2}{\gamma\mu}$ = units of length

Then $$\frac{1}{r(\phi)} = \frac{1}{c}(1 + \varepsilon\cos\phi)$$

or $$\boxed{r(\phi) = \frac{c}{1+\varepsilon\cos\phi}}$$

Bounded Orbits

Two important cases: If $\varepsilon < 1$, then the denominator is always greater than zero, and then $r(\phi)$ has a maximum and a minimum. (bounded) But if $\varepsilon = 1$ or $\varepsilon > 1$, then the denominator goes to zero at some angle, and $r \to \infty$ (unbounded)

For $\varepsilon < 1$,

$$r_{min} = \frac{c}{1+\varepsilon} \quad \text{and} \quad r_{max} = \frac{c}{1-\varepsilon}$$

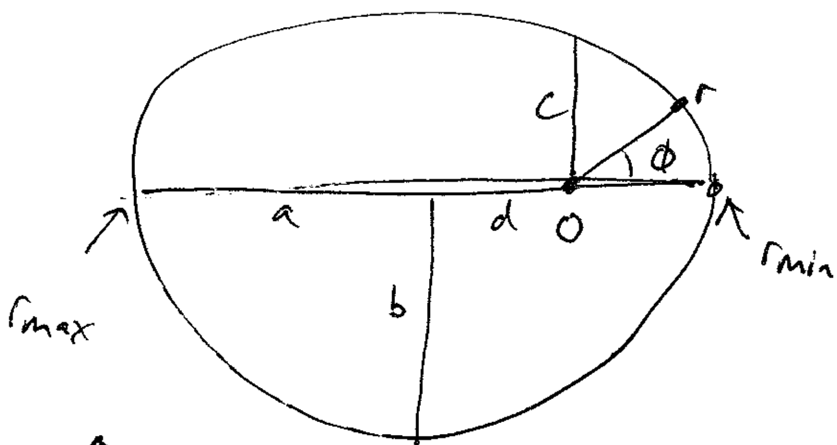
r_{min} is "perihelion" when $\phi = \emptyset$ and $\cos\phi = 1$.

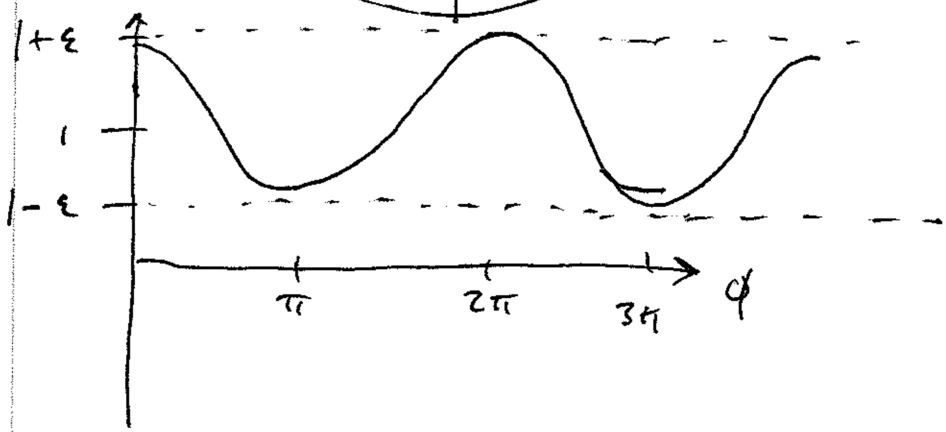
r_{max} is "aphelion" when $\phi = \pi$ and $\cos\phi = -1$.

Most importantly,

$$r(\phi=\emptyset) = r(\phi = 2\pi), \text{ so}$$

$r(\phi)$ is periodic with period 2π. In other words the orbits close on themselves after just one revolution.





The Equation $r(\phi) = \dfrac{c}{1+\varepsilon\cos\phi}$

can be rewritten in cartesian coordinates as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = \dfrac{c}{1-\varepsilon^2}$, $b = \dfrac{c}{\sqrt{1-\varepsilon^2}}$, and $d = a\varepsilon$.

This is the standard equation for an ellipse in cartesian coordinates.

The ratio of the major and minor axes is

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

or $$\boxed{\varepsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}}$$

This is the definition of the eccentricity of the ellipse. If $b=a$, the $\varepsilon = \emptyset$, and we have a circle with no eccentricity. If $b \gg a$, then $\varepsilon \to 1$, and we have high eccentricity.

Also $d = a\varepsilon$, $a\varepsilon$ is the distance from the center of the ellipse to the sun, or the focus of the ellipse.

Period: Kepler's 2nd Law follows from conservation of angular momentum:

$$\frac{dA}{dt} = \frac{\ell}{2\mu}$$

But the Area of an ellipse is πab, so the period (τ) is

$$\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell}$$

Square both sides and substitute $b^2 = a^2(1-\varepsilon^2)$

$$\tau^2 = \frac{4\pi^2 a^4 \overbrace{(1-\varepsilon^2)}^{c/a} \mu^2}{\ell^2} = \frac{4\pi^2 a^3 c \mu^2}{\ell^2}$$

And since $c = \frac{\ell^2}{\gamma\mu}$, we have

$$\tau^2 = \frac{4\pi^2 a^3 \mu}{\gamma}$$

For gravity, $\gamma = G M_s m_p = G\mu M$, $M = M_s + M_p \approx M_s$, where $M_s \gg m_p$ for the case of the sun and a planet, so

$$\gamma \approx G\mu M_s$$ Then

$$\boxed{\tau^2 = \frac{4\pi^2}{G M_s} a^3}$$ Kepler's 3rd Law.

Notice that for gravity, the mass of the planet (or comet or asteroid) has cancelled. So all objects in the same orbit have the same period.

Relationship between energy and eccentricity.

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)$$

Also $r_{min} = \frac{c}{1+\varepsilon} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$

Then $E = \frac{\gamma\mu(1+\varepsilon)}{2\ell^2}\left[\gamma(1+\varepsilon) - 2\gamma\right]$

$$\boxed{E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)}$$

Notice that $\varepsilon < 1$ gives a negative energy ($E<0$), so the orbit is <u>bounded</u>..

$\varepsilon = \underline{1}$ gives $E = \emptyset$, and unbounded orbit. (parabolic)
$\varepsilon > \underline{1}$ gives a positive energy, and an unbounded hyperbolic orbit. The lowest energy case for a given ℓ is $\varepsilon = \emptyset$, or $E = -\frac{\gamma^2\mu}{2\ell^2}$.

Apsidal angles and Precession

The r_{max} and r_{min} are also called "apsides." From energy considerations we can determine the angular change between any two values of r, and, in particular, any two apsides.

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$\therefore \quad \dot{r} = \sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{\ell^2}{\mu^2 r^2}}$$

substituting $\quad d\phi = \frac{d\phi}{dt}\frac{dt}{dr}dr = \frac{\dot{\phi}}{\dot{r}}dr$

Also, $\dot{\phi} = \frac{\ell}{\mu r^2}$, so

$$d\phi = \frac{\ell}{\mu r^2}\frac{1}{\dot{r}}dr$$

And putting in the above expression for $\dot{r}$:

$$d\phi = \frac{\ell}{\mu r^2}\frac{dr}{\sqrt{\frac{2}{\mu}\left(E-U(r)\right) - \frac{\ell^2}{\mu^2 r^2}}}$$

$$d\phi = \frac{\frac{\ell}{r^2}dr}{\sqrt{2\mu\left(E-U(r)\right) - \frac{\ell^2}{r^2}}}$$

$$d\phi = \frac{\frac{\ell}{r^2}dr}{\sqrt{2\mu\left(E-U(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

Now we can integrate between any two values of r to get the angular change:

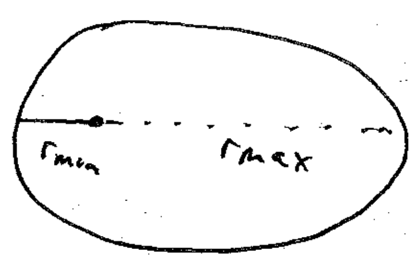
$$\phi(r) = \int_{r_1}^{r_2} \frac{\frac{\ell}{r^2}\, dr}{\sqrt{2\mu\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

In particular, if we integrate from r_{min} to r_{max}, and multiply by two, we get the angular change between two values of r_{min}:

$$r_{min} \rightarrow r_{max} \rightarrow r_{min}$$

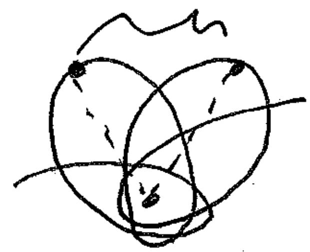
$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{\frac{\ell}{r^2}\, dr}{\sqrt{2\mu\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}}$$

For a closed elliptical orbit, $\Delta\phi$ will be 2π:



But depending on the force law, $\Delta\phi$ could be different than 2π:

Then we say that the orbit precesses:



or $u(r) \sim \frac{k}{r^n}$

It can be shown that for $F(r) \sim \frac{k}{r^{n+1}}$ where n is some integer, then closed paths occur when $n = -2$ or $+1$.

harmonic oscillator potential.

inverse square law potential

In general, $\cancel{\sqrt{n+2}}$ $\sqrt{-n+2}$ must be a rational number, for a closed path

Precession of Mercury

In the solar system, planets are subject to gravity of the sun, but also gravitational interactions with the other planets. The total force is <u>not</u> a simple inverse square law, and this causes the apsides of the elliptical orbits to precess. Newton argued that careful observation of the precession would provide a sensitive test of the inverse square law, once the effect of the other planets is included and accounted for.

For Mercury, calculations based on Newtonian inverse-square gravity, and including the effects of the other planets, predict that the precession should be <u>531"</u> (arc-seconds, $\frac{1}{3600}$ of a degree) <u>per century</u>. Observations found the Mercury's orbit precesses at <u>574"</u> per century, an excess of <u>43" per century</u>. This was first noticed in 1845 by Le Verrier. Similiar observations of Uranus' orbit led to the discovery of Neptune in 1846. In the case of Mercury, the 43" excess was finally understood to be a consequence of <u>General Relativity</u>, and this was one of the very first observational pieces of evidence in favor of GR..

In the case of a weak gravitational field, we can think of GR as being a small correction to Newton's Inverse Square Law of the form:

$$F(r) = \underbrace{-\frac{GMm}{r^2}}_{\text{Newton}} \underbrace{- \frac{3GM\ell^2}{c^2 m}\frac{1}{r^4}}_{\text{Einstein}}$$

Recall our Equation of Motion for $r(\phi)$:

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u^2(\phi)} F$$

when $u \equiv \frac{1}{r}$ and μ = reduced mass $\simeq m$

for the Sun - Mercury system.

or $$u''(\phi) + u(\phi) = \frac{m}{\ell^2 u^2(\phi)} F$$

In terms of u,

$$F(u) = -GMmu^2 - \frac{3GM\ell^2}{c^2 m} u^4$$

so we have

$$u''(\phi) + u(\phi) = \underbrace{\frac{GMm^2}{\ell^2}}_{\text{Newton}} + \underbrace{\frac{3GM}{c^2}}_{\text{Einstein.}} u^2(\phi)$$

Simplify notation: $\frac{1}{\alpha} \equiv \frac{GMm^2}{\ell^2}$

$$\delta = \frac{3GM}{c^2}$$

$$\boxed{u''(\phi) + u(\phi) = \frac{1}{\alpha} + \delta u^2(\phi)}$$

Non-linear equation of Motion. Solve by successive approximation. First guess: we take the pure elliptical orbit predicted by Newton alone:

$$u_1 = \frac{1}{\alpha}\left(1 + \varepsilon\cos\phi\right),$$ (where $\alpha = c$ from our earlier notation)

Then, substituting on the right hand side,

$$u''(\phi) + u(\phi) \approx \frac{1}{\alpha} + \frac{\delta}{\alpha^2}\left[1 + 2\varepsilon\cos\phi + \varepsilon^2\cos^2\phi\right]$$

$$= \frac{1}{\alpha} + \frac{\delta}{\alpha^2}\left[1 + 2\varepsilon\cos\phi + \frac{\varepsilon^2}{2}\left(1+\cos 2\phi\right)\right]$$

We must add a second function to our guess that reproduces this term in brackets

Such a function is

$$u_{particular}(\phi) = \frac{\delta}{\alpha^2}\left[\left(1 + \frac{\varepsilon^2}{2}\right) + \varepsilon\phi\sin(\phi) - \frac{\varepsilon^2}{6}\cos(2\phi)\right]$$

Thus our second guess is

$$u_2(\phi) = u_1(\phi) + u_{particular}(\phi)$$

$$u_2(\phi) = \left[\frac{1}{\alpha}(1+\varepsilon\cos\phi) + \frac{\delta\varepsilon}{\alpha^2}\phi\sin\phi\right]$$

$$+ \underbrace{\left[\frac{\delta}{\alpha^2}\left(1+\frac{\varepsilon^2}{2}\right) - \frac{\delta\varepsilon^2}{6\alpha^2}\cos 2\phi\right]}$$

These terms do not cause a precession, because they average out to a constant value. They represent a small, periodic disturbance to the normal Keplerian Ellipse.

The first term in brackets we call the "secular" terms:

$$U_{secular}(\phi) = \frac{1}{\alpha}\left[(1+\varepsilon\cos\phi) + \frac{\delta\varepsilon}{\alpha}\phi\sin\phi\right]$$

We can expand $1+\varepsilon\cos\left(\phi-\frac{\delta}{\alpha}\phi\right)$ as follows

$$1+\varepsilon\cos\left(\phi-\frac{\delta}{\alpha}\phi\right) = 1+\varepsilon\left(\cos\phi\cos\left(\frac{\delta}{\alpha}\phi\right)+\sin\phi\sin\left(\frac{\delta}{\alpha}\phi\right)\right)$$

$$\approx 1+\varepsilon\cos\phi + \frac{\delta\varepsilon}{\alpha}\phi\sin\phi$$

by approximating $\cos\left(\frac{\delta}{\alpha}\phi\right) \approx 1$, $\sin\left(\frac{\delta}{\alpha}\phi\right) \approx \frac{\delta}{\alpha}\phi$,

because δ is small.

Now we can write

$$u_{secular} \approx \frac{1}{\alpha}\left[1 + \varepsilon\cos\left(\phi - \frac{\delta}{\alpha}\phi\right)\right]$$

The apside precesses when the argument of the Cosine function increases by 2π:

$$\phi - \frac{\delta}{\alpha}\phi = 2\pi$$

or $$\phi = \frac{2\pi}{1-(\delta/\alpha)} \approx 2\pi\left(1+\frac{\delta}{\alpha}\right)$$

Therefore the effect of the Einstein term is that the apside is displaced in ϕ by an amount

$$\Delta \approx \frac{2\pi\delta}{\alpha} \text{ on each orbit.}$$

$$\approx 6\pi\left(\frac{GmM}{c\ell}\right)^2$$

$c = \alpha$

~~Usine~~ using ~~Be~~ a = semi-major axis $= \dfrac{c}{1-\varepsilon^2} = \dfrac{\alpha}{1-\varepsilon^2}$

$$a(1-\varepsilon^2) = c = \alpha$$
$$\approx \frac{\ell^2}{\mu\gamma}$$

we have $\ell^2 = \mu\gamma a(1-\varepsilon^2) = \mu(GMm)a(1-\varepsilon^2)$
$\approx m(GMm)a(1-\varepsilon^2)$

or $\ell^2 \approx GM m^2 a(1-\varepsilon^2)$

so $$\Delta = 6\pi \left(\frac{GmM}{c}\right)^2 \frac{1}{\ell^2}$$

$$\boxed{\Delta = \frac{6\pi GM}{ac^2(1-\varepsilon^2)}}$$

Δ is largest if a (semi-major axis) is small and ε is close to 1.

Mercury has the smallest (a) (0.387 au) and a reasonably large ε (0.2056), leading to

$$\Delta = 43.03'' \pm 0.03'' \text{ per century}$$

compared to observation of $43.11'' \pm 0.45''$ per century.

This was the 1st confirming evidence for general relativity.

Changes of Orbit

Orbital Equation: $r(\phi) = \dfrac{c}{1+\varepsilon\cos(\phi-\delta)}$

δ = phase at $\phi = \phi_0$ for any single orbit this can be chosen to be zero.

ε = eccentricity

$c = \dfrac{\ell^2}{\gamma\mu}$, $\gamma = GM_1M_2$ for gravity

μ = reduced mass

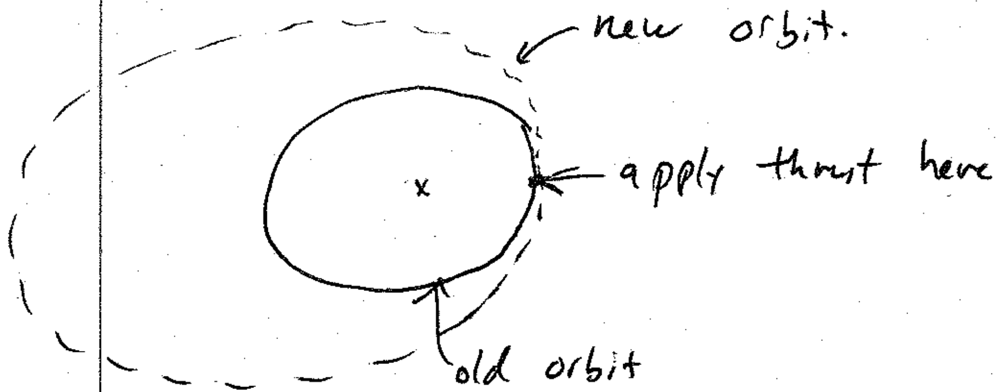
ℓ = angular momentum

E = total energy = $\dfrac{\gamma^2\mu}{2\ell^2}(\varepsilon^2-1)$

If we apply thrust for a short period of time, we can change ℓ (by changing $\vec{p}$ and $\vec{r}\times\vec{p}$) and E. The eccentricity ε will adjust so that

$$E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2-1)$$

is still satisfied. The new orbit will meet the old orbit at the location when the thrust was applied!



The angle at which the thrust is applied is very important. If applied along the direction of the center-of-force (the sun or earth), ~~the~~ and if we choose that location as our origin, then the angular momentum cannot change, but E will.

Or, if applied tangentially to the orbit, then both l and E will change. ε will adjust to satisfy the new l & E.

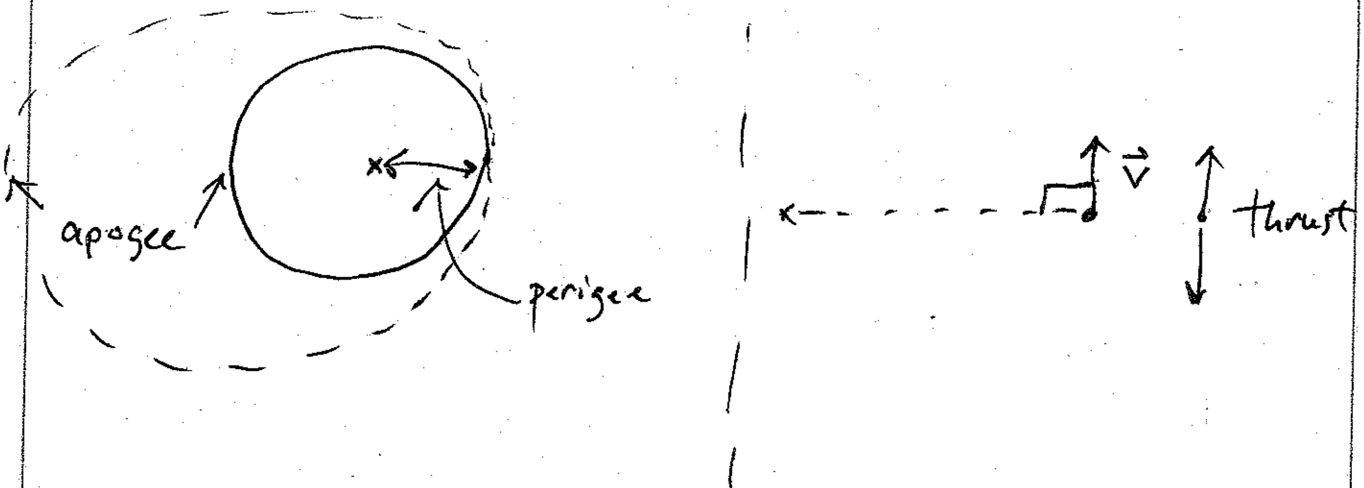
For the case of a short burst of thrust, we can set the two orbital equations equal to each other at the location of the thrust:

$$\frac{c_1}{1+\varepsilon_1\cos(\phi_0-\delta_1)} = \frac{c_2}{1+\varepsilon_2\cos(\phi_0-\delta_2)}$$

where ϕ_0 is the location of the thrust, and $c_1, \varepsilon_1, \delta_1$, and $c_2, \varepsilon_2, \delta_2$ are the orbital parameters. Given a known change in velocity, we can calculate the new E_2 by adding KE to the old E_1, and the new l_2 from the old l_1 (adding $\vec{r} \times m\Delta\vec{v}$). This gives us the new c_2 and ε_2. Then the phase δ_2 can be determined from the above equation (perhaps tediously).

Special case: A tangential thrust applied at Perigee

Perigee: position of closest approach.



Choose ϕ_0 such that $\delta_1 = \phi$. Then perigee occurs when $\cos(\phi_0 - \delta_1) = +1$

and $\cos(\phi_0 - \delta_2) = +1$ so

$$\left.\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2}\right\} \text{distances of closest approach.}$$

Let λ be the ratio of velocities (after thrust and before.):

$$v_2 = \lambda v_1 \qquad \lambda > 1 \text{ means forward thrust}$$
$$0 < \lambda < 1 \text{ means backward thrust}$$

At perigee angular momentum is very simple:

$$\ell = \mu r v$$

so $\ell_2 = \lambda \ell_1$

while $c_2 = \dfrac{\ell_2^2}{\gamma\mu}$ so $c_2 = \lambda^2 c_1$. Then our orbital equality is

$$\frac{c_1}{1+\varepsilon_1} = \frac{\lambda^2 c_1}{1+\varepsilon_2}$$

$$1 + \varepsilon_2 = \lambda^2(1+\varepsilon_1) = \lambda^2 + \lambda^2\varepsilon_1$$

$$\boxed{\varepsilon_2 = \lambda^2\varepsilon_1 + \lambda^2 - 1}$$

For $\lambda > 1$, (forward thrust), $\varepsilon_2 > \varepsilon_1$ so the new orbit has the same perigee, but higher eccentricity (and higher E & ℓ).

For $\lambda < 1$, (backward thrust), $\varepsilon_2 < \varepsilon_1$, and the new orbit lies inside the old orbit and is closer to circular.