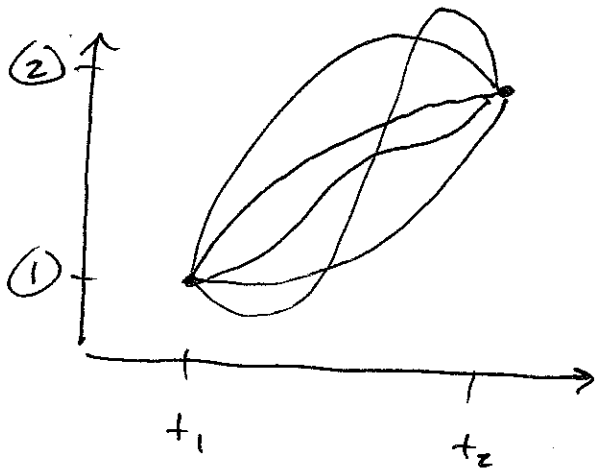
Hamilton's Principle, Stated imprecisely

Imagine a single particle traveling from location (1) to location (2) under the influence of a conservative force. It is located at (1) at time t_1, and at (2) at time t_2. Now imagine all ~~possible~~ mathematically possible ways to get from (1) to (2), subject to the condition that we are at (1) at t_1 and at (2) at t_2. In one dimension we can draw a diagram:



⇐ Lots of different mathematically possible ways to get from (1) to (2) while time advances from t_1 to t_2.

~~Hamilton's Principle says that~~ Nature chooses just one of these mathematically possible paths. It is the only and only path which is described by classical mechanics. (Newton's Laws)

Hamilton's Principle says (roughly) that the path chosen by nature is the one which minimizes the difference between the time averaged Kinetic

Energy and time-averaged Potential Energy:

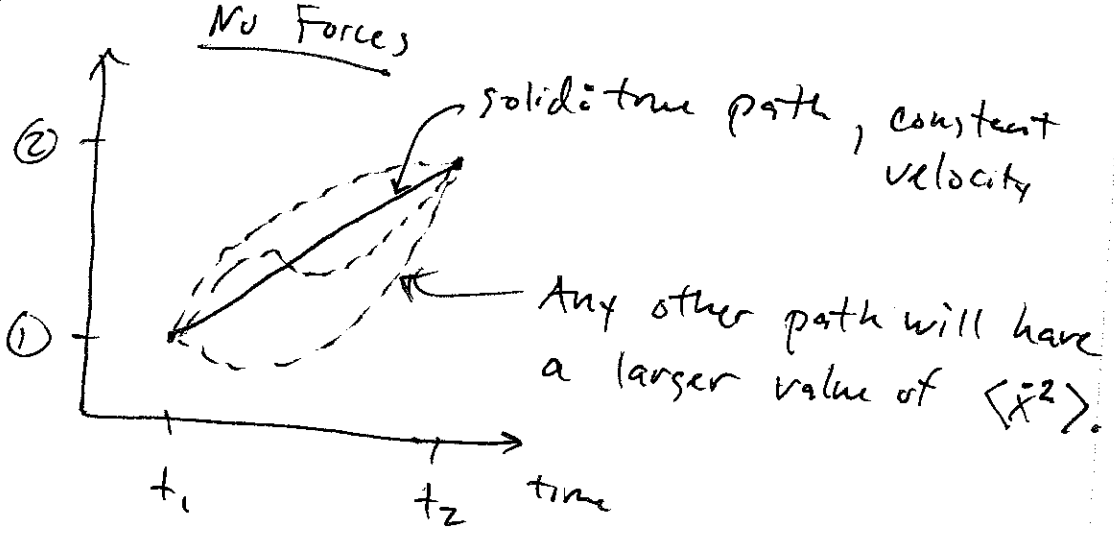
True path minimizes $\langle T \rangle - \langle U \rangle$

($\langle T \rangle$: time average; $\langle U \rangle$: time average)

A simple example: One dimensional particle with no forces:

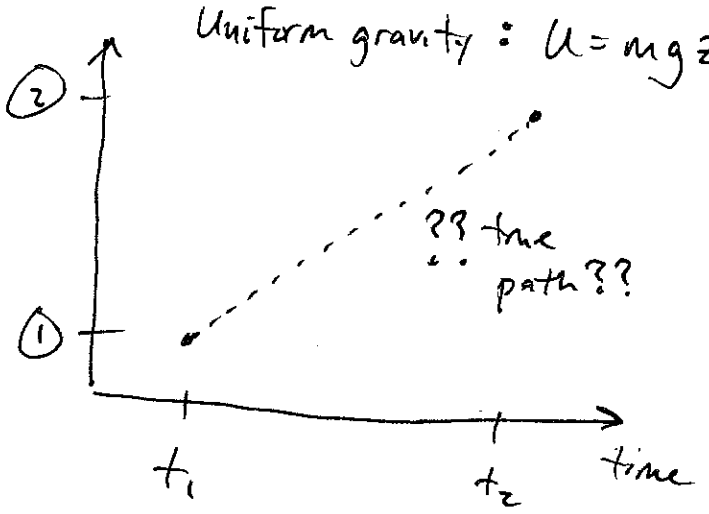
Since there are no forces, $\langle U \rangle = 0$. So Hamilton's principle says the true path minimizes the time-averaged KE: $\langle T \rangle$.

$\langle T \rangle = \frac{1}{2} m \langle \dot{x}^2 \rangle$ What is the best way to minimize $\langle \dot{x}^2 \rangle$? Answer: The particle should travel at the average velocity for the entire path. Any deviation from the average velocity will have a penalty in $\langle \dot{x}^2 \rangle$, making it larger.



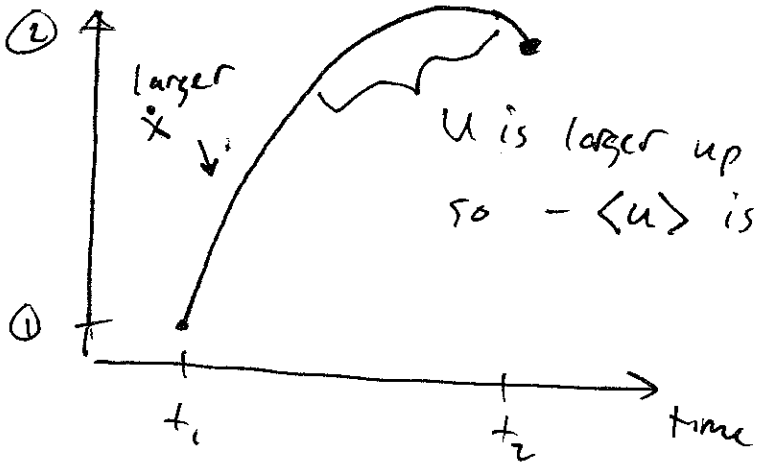
Of course for the simple case of no forces, we know that constant velocity motion is correct (it is consistent with Newton's Laws and Observations).

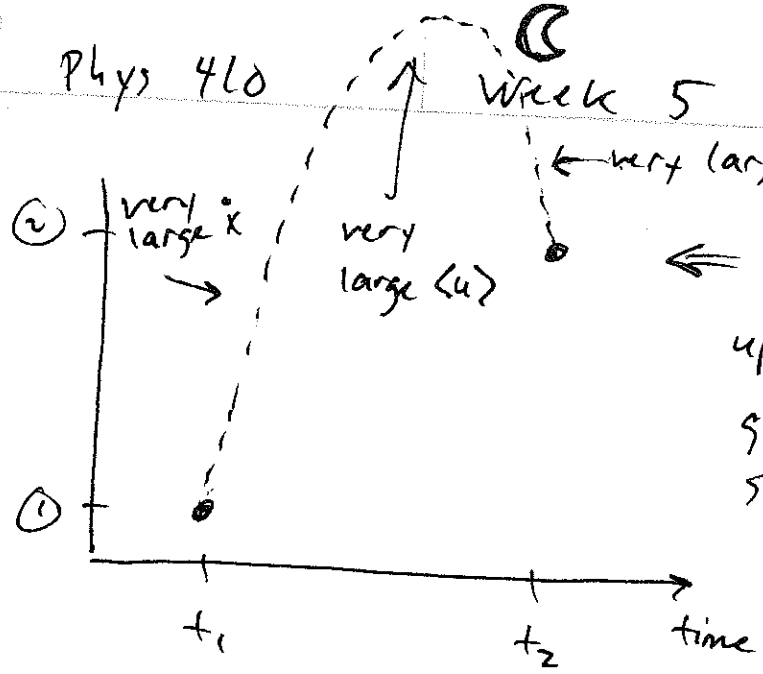
A 2nd simple example: Ball thrown upward in uniform gravitation field



Does the ball still want to travel at uniform, constant velocity?

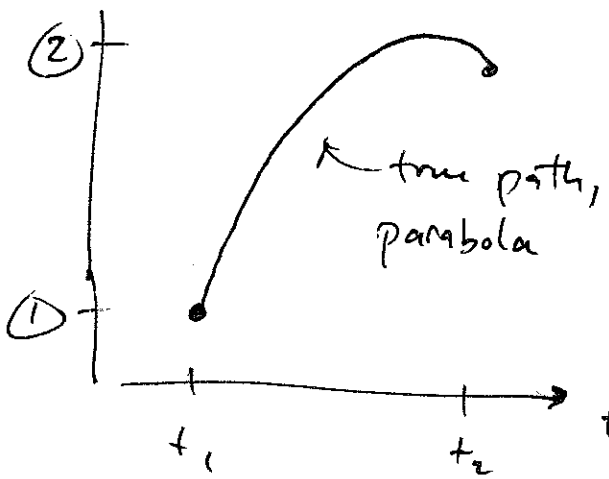
No, in this case we can get a smaller $\langle T \rangle - \langle U \rangle$ by taking advantage of the larger $\langle U \rangle$ that happens when the ball is higher:





No, it can't go all the way to the moon and still minimize $\langle T \rangle - \langle u \rangle$. The problem is that it must reach point ② by time t_2. If it goes all the way to the moon and back in the available time, it will need a huge velocity. Then $\langle T \rangle$ will be very large and this will make $\langle T \rangle - \langle u \rangle$ larger than necessary.

It turns out that the minimal $\langle T \rangle - \langle u \rangle$ occurs for a parabolic path (and we know from Newton that the parabolic path is the correct one.) From the perspective of Hamilton's Principle, the parabola is a compromise



between ~~need~~ needing a ~~large~~ very negative $-\langle U\rangle$ and not having a too large $\langle T\rangle$.

The basic point is that we need to reach a large height quickly to get $-\langle U\rangle$ very negative, but reaching it too quickly causes $\langle T\rangle$ to be too large. The true path balances these two requirements

More Precise Statement of Hamilton's Principle

The time average of $\langle T\rangle - \langle U\rangle$ can be written mathematically as

$$\int_{t_1}^{t_2} (T-U)\,dt$$

We call this quantity "the Action"

$$\text{Action} \equiv S \equiv \int_{t_1}^{t_2} (T-U)\,dt = \text{units of Energy} \times \text{time} = \text{J}\cdot\text{seconds in SI.}$$

Strictly speaking the time average energy difference should be $\frac{S}{\Delta t}$, however the Δt is not needed to get the physics correct; so we will not include it or mention it again.

In general T and U are functions of $x, \dot{x}$, and t:

$$T(\dot{x}, x, t) - U(x, t)$$ (U depends on x, t, but not $\dot{x}$)

Of course this is what we call the Lagrangian.

$$\mathcal{L}(\dot{x}, x, t) \equiv T - U$$

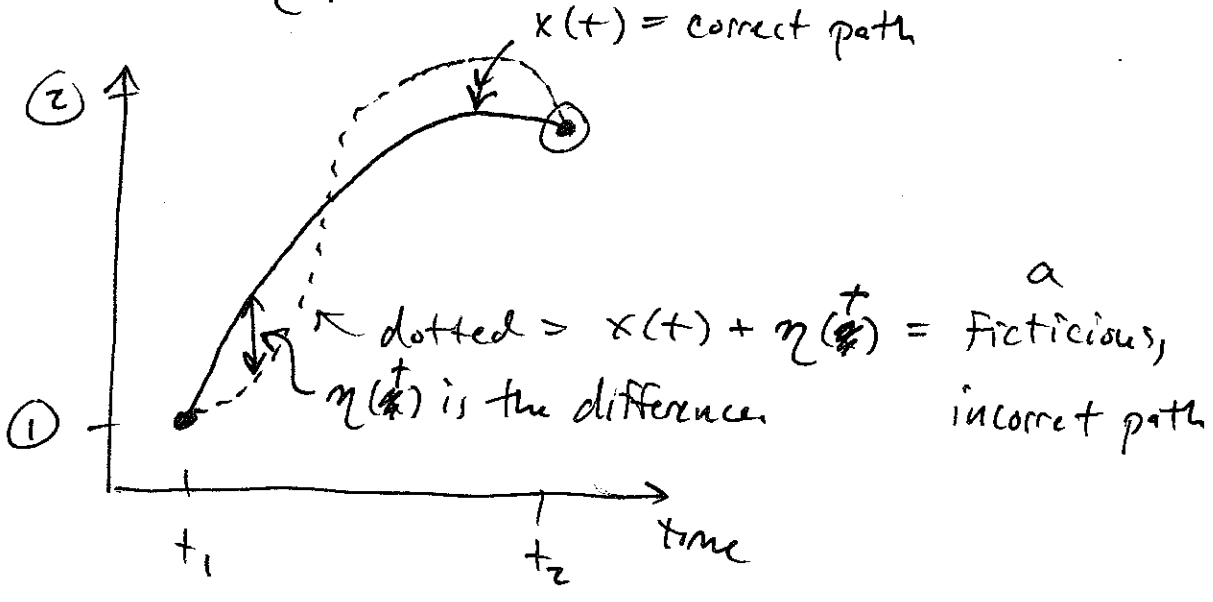
So the Action is defined as

$$S \equiv \int_{t_1}^{t_2} \mathcal{L}(\dot{x}, x, t)\, dt$$

To precisely state Hamilton's Principle, we must define the concept of "Stationary Action".

Stationary Action

Let the true, correct path as realized in nature be called $x(t)$. We will consider deviations from this path. Let the deviations be called $\eta(t)$:



We will consider all possible deviation functions $\eta(t)$, except that $\eta(t=t_1) = 0$ and $\eta(t=t_2) = 0$. The reason is that we require the particle to be located at (1) at t_1 and at (2) at t_2, so the deviation function must be zero at t_1 & t_2.

One modification to our expression for the fictitious path: Let α be a small, dimensionless parameter. Then we let our fictitious path be written

$$X(t) = x(t) + \alpha\eta(t)$$

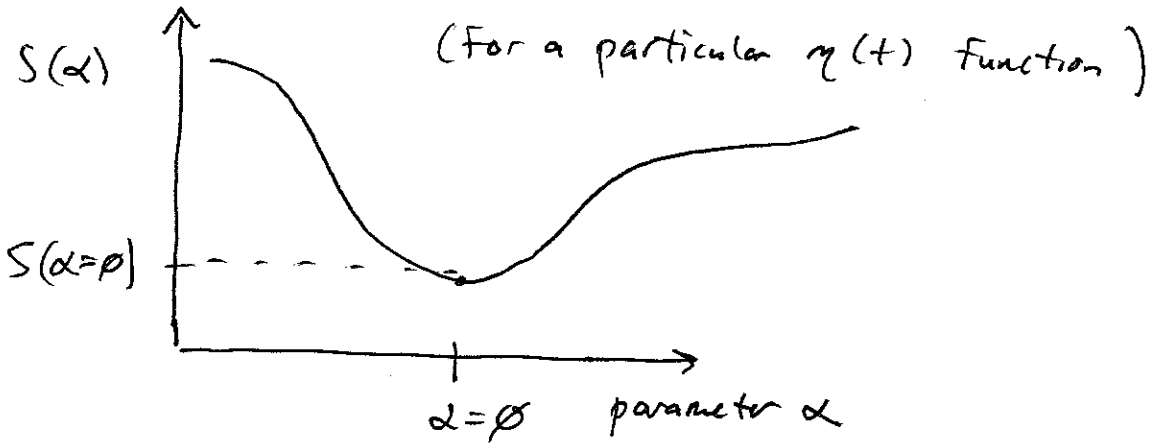
- X: capital x = fictitious
- x: true path
- α: small parameter
- η: deviation function

For any deviation function $\eta(t)$, we will consider how the Action depends on the small parameter α:

$$S(\alpha) = \text{Action for value } \alpha = \int_{t_1}^{t_2} L(X, \dot{X}, t)\, dt$$

$$= \int_{t_1}^{t_2} L(\dot{x}+\alpha\dot{\eta}, x+\alpha\eta, t)\, dt$$

If you choose a particular deviation function $\eta(t)$, and you allow α to vary, in general the action S will vary:



S will vary as some polynomial in α (expanding about $\alpha=0$: $S(\alpha) = S_0 + \alpha \frac{dS}{d\alpha} + \frac{1}{2}\alpha^2 \frac{d^2S}{d\alpha^2} + \frac{1}{6}\alpha^3 \frac{d^3S}{d\alpha^3} + \cdots$

Definition of "Stationary Action": If S has no first-order dependence on α, we say that the Action is stationary.

Notice that a local minimum or a local maximum or a saddle point counts as stationary. Also, as we will see, the exact form of the deviation function $\eta(t)$ makes no difference in terms of locating the stationary action points.

Finally we can state Hamilton's Principle precisely:

> The actual path followed between points (1) and (2) from time t_1 to t_2 will be such that the Action integral is stationary:
>
> $$S = \int_{t_1}^{t_2} \mathcal{L}\, dt \quad \Leftarrow \text{stationary for the true path.}$$

This is also known as the principle of stationary action. The Euler-Lagrange Equation of Motion from Lagrangian Mechanics is a consequence of Hamilton's Principle.

Note: Sometimes this is called the "principle of least action." It is true that the stationary point is often a local minimum, however sometimes it is a saddle point; not a minimum. So we should stick with "stationary action" rather than "least action".

Note Also: The true path is never a local maximum of S. Saddle points and local minima are the two possibilities. We will see why local maxima never occur.

δ - notation

To indicate that the Action should be stationary, we sometimes use the δ - notation:

$$\delta S = \text{first-order variation of } S \text{ with } \alpha = 0$$

or $$\delta\left[\int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\, dt\right] = 0$$

Derivation of the Euler-Lagrange Equation of Motion from Hamilton's Principle:

It is rather easy to show that Hamilton's Principle leads to the Euler-Lagrange Equation of Motion. The basic trick is to integrate by parts!

Recall that

$$S(\alpha) = \int_{t_1}^{t_2} \mathcal{L}(X, \dot{X}, t)\, dt$$

where $$X(t) = x(t) + \alpha \eta(t)$$

- $X(t)$: fictitious path
- $x(t)$: true path
- α: small parameter
- $\eta(t)$: deviation Function.

Hamilton's Principle requires that

$$\frac{dS}{d\alpha} = 0 \Leftarrow \text{no } 1^{st} \text{ order dependence on } \alpha.$$

So we have

$$\frac{dS}{d\alpha} = \frac{\partial}{\partial\alpha}\int_{t_1}^{t_2} \mathcal{L}(X, \dot{X}, t)\, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial\alpha}\left(\mathcal{L}(X, \dot{X}, t)\right) dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial\alpha}\left[\mathcal{L}(x+\alpha\eta, \dot{x}+\alpha\dot{\eta}, t)\right] dt$$

(chain rule on each argument)

$$= \int_{t_1}^{t_2}\left[\frac{\partial\mathcal{L}}{\partial x}\underbrace{\frac{\partial(x+\alpha\eta)}{\partial\alpha}}_{\eta} + \frac{\partial\mathcal{L}}{\partial\dot{x}}\underbrace{\frac{\partial(\dot{x}+\alpha\dot{\eta})}{\partial\alpha}}_{\dot{\eta}}\right]$$

$$= \int_{t_1}^{t_2}\left[\frac{\partial\mathcal{L}}{\partial x}\eta + \frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{\eta}\right] dt = 0 \quad \text{by Hamilton's Principle}$$

This must hold true for any deviation function $\eta(t)$.

Now we can integrate the second term by parts:

$$\int_{t_1}^{t_2} \frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{\eta}\, dt = \left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\eta\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta\,\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) dt$$

The good news is that the deviation function $\eta(t)$ is defined to be zero at the two endpoints:

$$\eta(t=t_1) = \phi \quad \text{and} \quad \eta(t=t_2) = \phi$$

This is because η is the deviation from the true path, and we require that deviation to be zero at the endpoints.

Therefore

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\,\eta\right]_{t_1}^{t_2} = \phi$$

So $$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{x}}\,\dot{\eta}\,dt = -\int_{t_1}^{t_2} \eta \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) dt$$

And then Hamilton's Principle is

$$\frac{dS}{d\alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial x}\eta - \eta \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right] dt = \phi$$

$$\frac{dS}{d\alpha} = \int_{t_1}^{t_2} \eta \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right] dt = \phi$$

This integral must be true for any function η (satisfying the boundary condition $\eta(t=t_1) = \phi = \eta(t=t_2)$.) Therefore the factor in brackets must be zero:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \phi$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}}$$

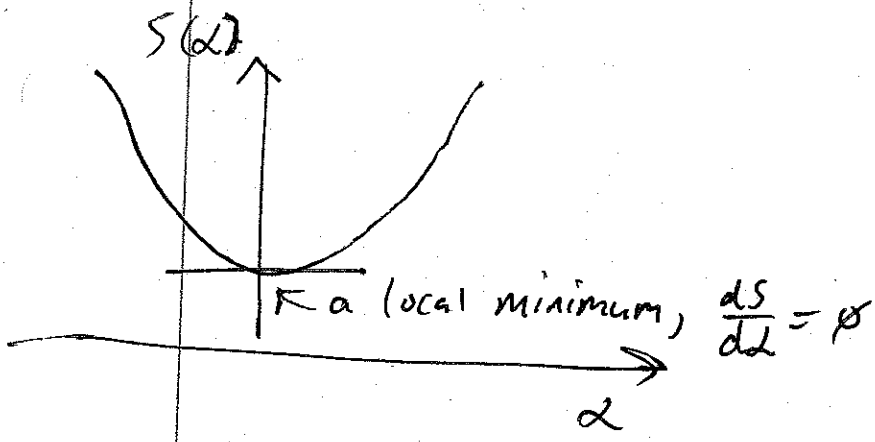
Euler-Lagrange Eq. of Motion.

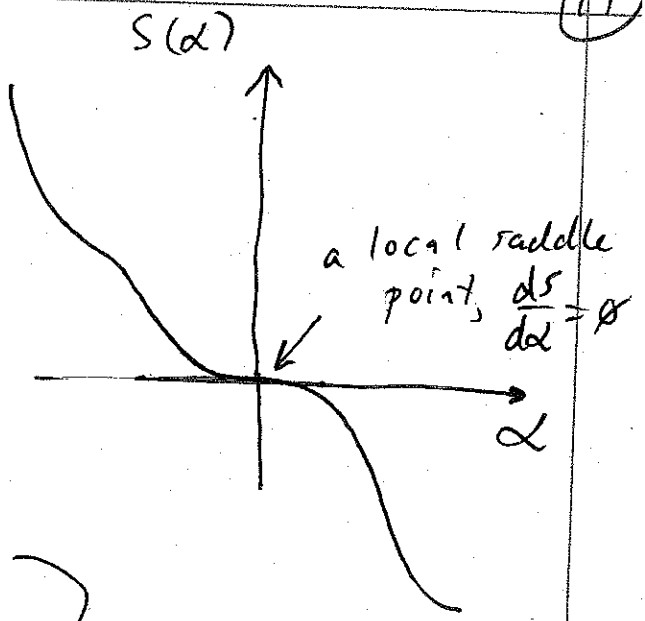
Therefore, to find the path that satisfies Hamilton's Principle, we should require that the Euler-Lagrange Equation of Motion holds true. Conversely, any path satisfying the E-L Equation will necessarily have a stationary action (no 1st order dependence on the variation function α).

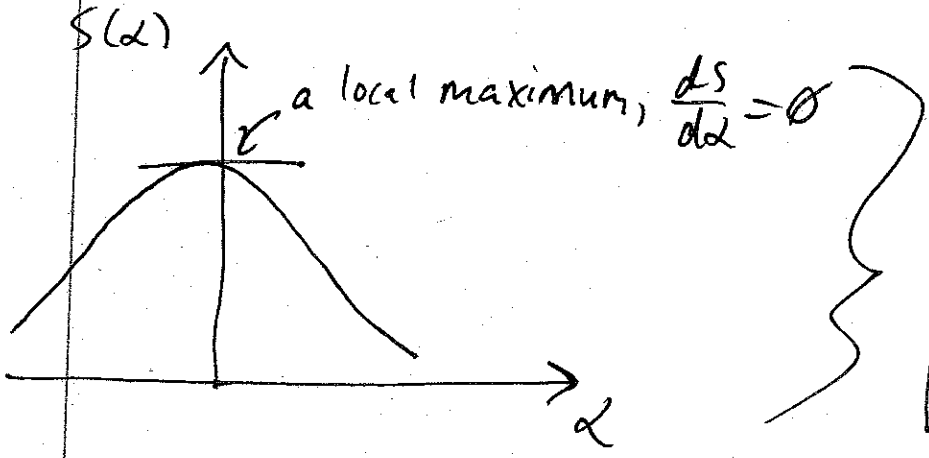
A minimum, a maximum, or a saddle point?

In principle you can find the action to be stationary wherever $\frac{dS}{d\alpha} = 0$, that is, wherever there is no first-order dependence of the action on the small parameter α. This means that local minima, local maxima, and local saddle points could be candidates for stationary action. In practice, however, for the action of classical Mechanics, the stationary points will never be local maxima:

Illustration: choose a particular deviation function $\eta(t)$; and sketch the action as a function of α

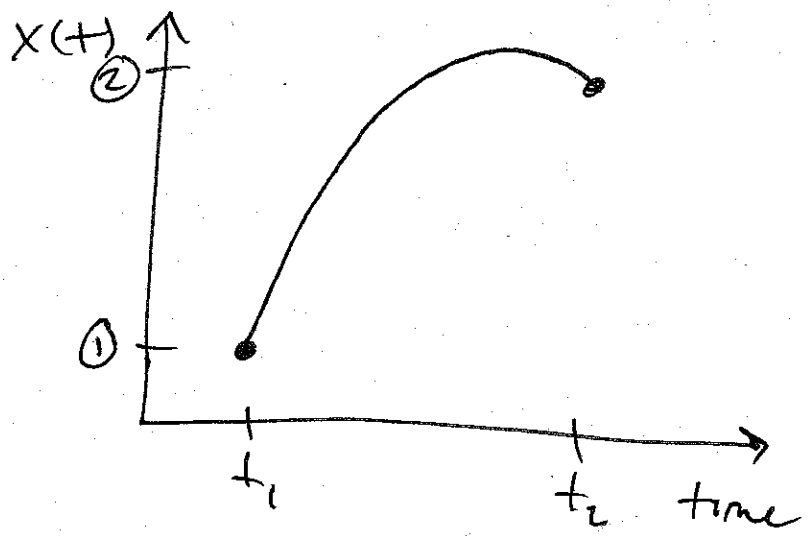




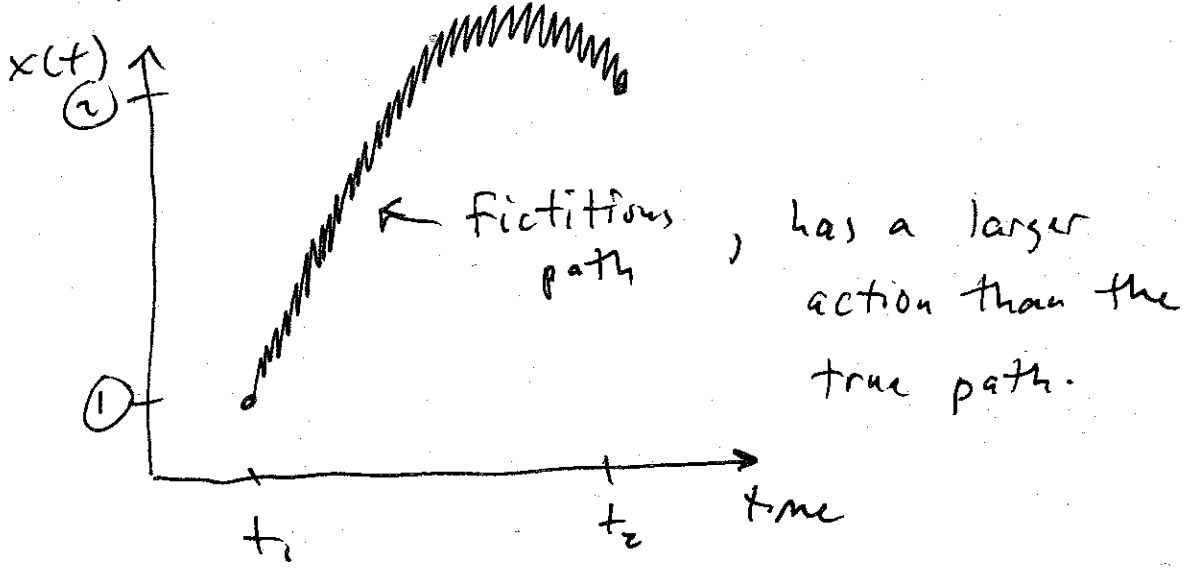


Does not occur for classical mechanics.

It's easy to see that the correct, true path is never a local maximum because we can always find nearby paths with more KE, leading to a larger action. For example, suppose the true path is a parabola



We can add a fictitious high-frequency, low amplitude oscillation to this path:



, has a larger action than the true path.

For this fictitious path, the KE will be <u>much larger</u> on average. However, the average potential energy can be very similar to the original potential energy, because the amplitude of oscillation is small, and because the time average will smooth out the average value. Therefore $\langle T \rangle - \langle U \rangle$ will be larger than that for the true path.

<u>Conclusion</u>

The true path will have either a local minimum or a saddle point in the action, but never a local maximum.

Chain Rule Details

X is a function of x & α: $X = X(x,\alpha) = x + \alpha\eta$

We consider η to be a fixed parameter, not a variable. Conceptually, we pick one and only one η function at a time, and we consider the minimization of the action within this limited context.

Derivatives: $\frac{\partial X}{\partial x} = 1$, $\boxed{\frac{\partial X}{\partial \alpha} = \eta}$, also $\frac{\partial x}{\partial X} = 1$

Then $\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial \mathcal{L}}{\partial x}\underbrace{\frac{\partial x}{\partial X}}_{1} = \boxed{\frac{\partial \mathcal{L}}{\partial x}}$

Also: $\dot{X} = \dot{X}(\dot{x},\alpha) = \dot{x} + \alpha\dot{\eta}$

Derivatives: $\frac{\partial \dot{X}}{\partial \dot{x}} = 1$, $\boxed{\frac{\partial \dot{X}}{\partial \alpha} = \dot{\eta}}$, also $\frac{\partial \dot{x}}{\partial \dot{X}} = 1$

Then $\frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{\partial \mathcal{L}}{\partial \dot{x}}\underbrace{\frac{\partial \dot{x}}{\partial \dot{X}}}_{1} = \boxed{\frac{\partial \mathcal{L}}{\partial \dot{x}}}$

Finally, the multivariable chain rule says

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathcal{L}(\dot{X}, X, t)}{\partial \alpha} = \frac{\partial \mathcal{L}}{\partial X}\frac{\partial X}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{X}}\frac{\partial \dot{X}}{\partial \alpha}$$

$$\boxed{= \frac{\partial \mathcal{L}}{\partial x}\eta + \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\eta}}$$

Three Important Consequences of Hamilton's Principle

1) In more than one dimension, described by more than one coordinate, we get the Euler-Lagrange Eq. of Motion for each coordinate.

Partial proof: Let x & y be the coordinates. Then vary both x & y around the true path:

$$X(t) = x(t) + \alpha\,\xi(t) \quad , \quad Y(t) = y(t) + \beta\,\eta(t).$$

For the action to be stationary, we require

$$\frac{\partial S}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial S}{\partial \beta} = 0.$$

It is straightforward to show (using the same steps used previously (integration by parts), that these requirements lead to 2 Euler-Lagrange Equations:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

2) Non-cartesian and non-orthogonal coordinates.

What happens if we choose to use a non-orthogonal coordinate system? For example, let

$$q_i = q_i(\vec{r}) \quad , \; i = 1, 2, 3$$ (with 3 degrees of freedom)

and conversely

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

In the non-orthogonal system,

$$\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

Hamilton Principle says

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\, dt = 0$$

But the value of an integral does not depend on the variables used to describe the integrand. We are free to make any change of variables we like, and the correct path will still be stationary, even in our new coordinate system. So the Euler-Lagrange equations must hold true in the q_i system:

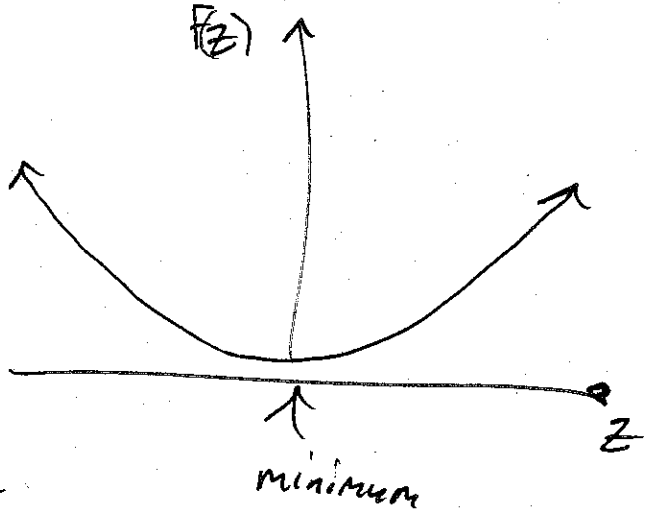
$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

We can make an analogy with ordinary differential calculus. Consider a function

$$f(z) = z^2$$

minimum at $z = 0$;

$$\frac{df}{dz}(z=0) = 0.$$



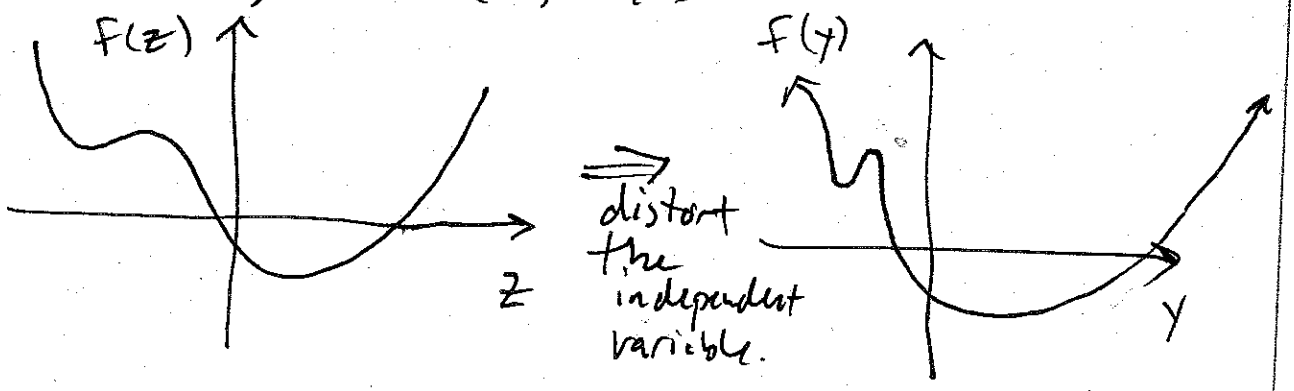
Now change variables: let

$$z = y^4,$$

Then $f(y) = y^8$. Still f has a minimum at $z = 0$ and $y = 0$:

$$\frac{df}{dy}(y=0) = 0.$$ So f' is zero in both coordinate system.

More intuitively, a change of variable in ordinary calculus is like stretching or compressing the (x) axis



Even with the new (x) axis, the local minima still correspond to each other. But note that this is only true for the points where $F' = \emptyset$:

IF $F(z) = z^2$ again and $z = y^4$ so $F(y) = y^8$, then $F' = \emptyset$ when $z = 0$ and $F' = \emptyset$ when $y = \emptyset$.

But what about $F' = 1$? Is this the same in both coordinate systems? Try it

In z: $F' = 1$ when $z = \frac{1}{2}$.

In y: $F' = 1$ when $y = \left(\frac{1}{8}\right)^{1/7}$.

But $z = \frac{1}{2}$ does not correspond to $y = \left(\frac{1}{8}\right)^{\frac{1}{7}}$

$z = \frac{1}{2} \Rightarrow y = \left(\frac{1}{2}\right)^4$, not $y = \left(\frac{1}{8}\right)^{\frac{1}{7}}$.

So $F' = 1$ is not a coordinate independent statement, unlike $F' = \emptyset$, which is coordinate independent.

3) (Important Consequence #3) Proof of Lagrange's Eq. of Motion for constrained systems.

Let a particle move on a constrained surface:

$$\vec{R}(t) = \vec{r}(t) + \vec{\xi}(t)$$

- $\vec{R}(t)$: Fictitious path, still constrained to the same surface
- $\vec{r}(t)$: Correct path, on a constrained surface
- $\vec{\xi}(t)$: Fictitious deviation from the true path, also constrained to the same surface

Now we can show that the action is stationary when $\vec{\xi}(t) = 0$, so Hamilton's Principle chooses the correct path.

Proof: Let $\delta S \equiv S - S_0$, the change in the action due to a first order deviation $\vec{\xi}$. Then δS is due to a change in the Lagrangian:

$$\delta \mathcal{L} = \mathcal{L}(\vec{R}, \dot{\vec{R}}, t) - \mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$

Using $\vec{R} = \vec{r} + \vec{\xi}$, and $\mathcal{L}(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, t)$,

we have $\delta \mathcal{L} = \frac{1}{2} m \left[(\dot{\vec{r}} + \dot{\vec{\xi}})^2 - \dot{\vec{r}}^2 \right] - \left[U(\vec{r} + \vec{\xi}, t) - U(\vec{r}, t) \right]$

or $\delta\mathcal{L} = m\dot{\vec{r}}\cdot\dot{\vec{\varepsilon}} - \vec{\varepsilon}\cdot\vec{\nabla}U + \underbrace{\mathcal{O}(\varepsilon^2)}_{\text{2nd order terms}}$

Now the change in the action is

$$\delta S = \int_{t_1}^{t_2} \delta\mathcal{L}\, dt = \int_{t_1}^{t_2} \left[\underbrace{m\dot{\vec{r}}\cdot\dot{\vec{\varepsilon}}}_{\text{integrate this by parts.}} - \vec{\varepsilon}\cdot\vec{\nabla}U\right] dt$$

$$\Longrightarrow \int_{t_1}^{t_2} m\dot{\vec{r}}\cdot\dot{\vec{\varepsilon}}\, dt = \underbrace{\left(m\dot{\vec{r}}\cdot\vec{\varepsilon}\right)_{t_1}^{t_2}}_{\text{zero, because } \vec{\varepsilon} = \emptyset \text{ at } t_1 \text{ \& } t_2 \text{ by definition.}} - \int_{t_1}^{t_2} m\ddot{\vec{r}}\cdot\vec{\varepsilon}\, dt$$

So $$\delta S = \int_{t_1}^{t_2} \left[-m\ddot{\vec{r}}\cdot\vec{\varepsilon} - \vec{\varepsilon}\cdot\vec{\nabla}U\right] dt$$

$$= -\int_{t_1}^{t_2} \vec{\varepsilon}\cdot\underbrace{\left[m\ddot{\vec{r}} + \vec{\nabla}U\right]}_{\text{Since } \vec{r} \text{ is the } \underline{\text{true path}}, \; m\ddot{\vec{r}} \text{ is equal to the } \underline{\text{total}} \; \underline{\text{Force}} \text{ on the particle.}} dt$$

$$\vec{F}_{\text{total}} = \vec{F}_{\text{constraint}} + \vec{F}_{\text{conservative}} = m\ddot{\vec{r}}$$

Also, $\vec{F}_{conservative} = -\vec{\nabla}U$, so

$$m\ddot{\vec{r}} + \vec{\nabla}U = \left(\vec{F}_{constraint} + \vec{F}_{conservative}\right) + \vec{\nabla}U$$

$$= \left(\vec{F}_{constraint} - \vec{\nabla}U\right) + \vec{\nabla}U$$

$$= F_{constraint}.$$

So $$\delta S = -\int_{t_1}^{t_2} \vec{\xi} \cdot \vec{F}_{constraint}\, dt$$

But the Forces of constraint are normal to the surface of constraint, while $\vec{\xi}$ is a deviation in the plane of constraint. So

$$\vec{\xi} \cdot \vec{F}_{constraint} = 0 \Leftarrow \text{They are perpendicular.}$$

So we have shown that the constrained system satisfies

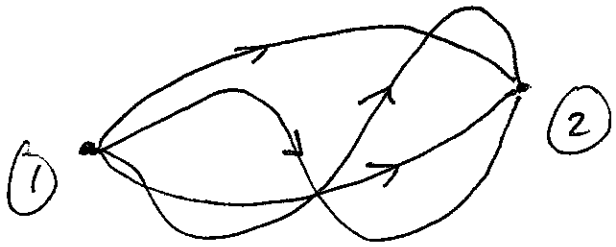
$$\boxed{\delta S = 0,} \Leftarrow \boxed{\text{constrained system satisfies Hamilton's Principle.}}$$

which means that we are justified in using the Euler-Lagrange Equation of motion even in the presence of constraints.

Hamilton's Principle & Quantum Mechanics

Hamilton's Principle has a profound justification in Quantum Mechanics. This is most easily seen in Feynman's Path Integral formulation of QM. The path integral formulation is not usually taught in undergraduate quantum mechanics, but it is completely equivalent to the schroedinger picture.

According to the path integral formalism, you can calculate the probability for a quantum mechanical particle to travel from (1) to (2) by considering all possible paths:



Every mathematically possible path gets an amplitude

$$e^{iS/\hbar}$$

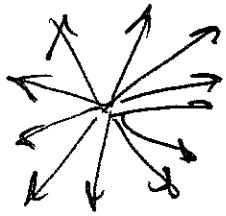
where S is the action for each path. To get the total probability you must sum up all the amplitudes from all the paths and square them:

$$\text{Prob}(1\to 2) = \left| e^{iS_1/\hbar} + e^{iS_2/\hbar} + e^{iS_3/\hbar} + e^{iS_4/\hbar} + \ldots \right|^2$$

(of course, there are usually an infinite number of paths, so you must really integrate.)

How does this picture justify Hamilton's Principle?

Answer: If a particular path has neighboring paths where the action rapidly changes between them, then the phases $e^{iS/\hbar}$ for those paths will generally cancel out and add up to nothing:

Sum of:  = • (not much)

However, if a particular path is stationary, then the nearby paths will have very similar actions, because, by definition, stationary paths have no first-order dependence on the size of the variation. So for a stationary path, all the nearby paths add up coherently and produce a large probability:

Sum of: 

So in the classical limit, only the stationary paths are important, and we can find the true path by requiring that it be stationary.

The Calculus of Variations

The formalism of Hamilton's Principle is an example of a branch of mathematics known as the calculus of variations. In the calculus of variations we assign a number to each function. We then allow the function to vary, and we ask "what function makes the number a minimum, maximum, or a saddle point?"

Note the difference with ordinary analysis:

- A function assigns a number to every number:

$$x(t)$$

- A "functional" assigns a number to every function:

$$S[x(t)]$$

The Action of classical Mechanics is a functional. We can transcribe most of our results from Hamilton's Principle to use the calculus of variations for other problems.

Suppose we have a functional

$$S = \int_{x_1}^{x_2} F[y(x), y'(x), x]\, dx$$

where $y(x)$ is an unknown function and $y'(x)$ is its derivative.

We wish to find $y(x)$ such that S is a minimum (or maximum, or stationary.). How should we find the correct $y(x)$?

We've already solved this problem for the case that $F = \mathcal{L}$, the Lagrangian. The answer was that the correct function should satisfy the Euler-Lagrange Equation. But the proof did not use the explicit form of the Lagrangian anywhere. (We never used the fact that $\mathcal{L} = T - U$, for example.) So the same proof will work for any function F.

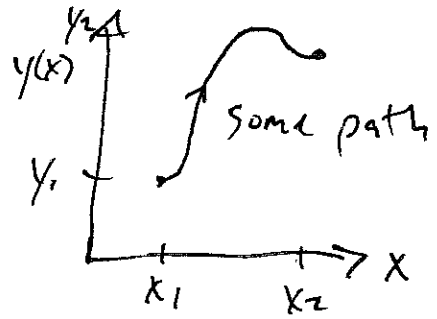
Therefore the answer to the variational problem

$$\delta\left[\int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx\right] = 0$$

is

$$\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}}$$

Trivial Ex: What is the shortest distance between 2 points?



The length of a short segment is

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

or $ds = \sqrt{1+(y'(x))^2}\,dx$

The total path length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}\,dx$$

We wish to minimize L. Therefore the integrand function must satisfy the Euler-Lagrange equation:

$$F(y, y', x) = \sqrt{1+(y')^2}$$

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

so $$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\frac{y'}{\sqrt{1+(y')^2}} = \text{constant} = c$$

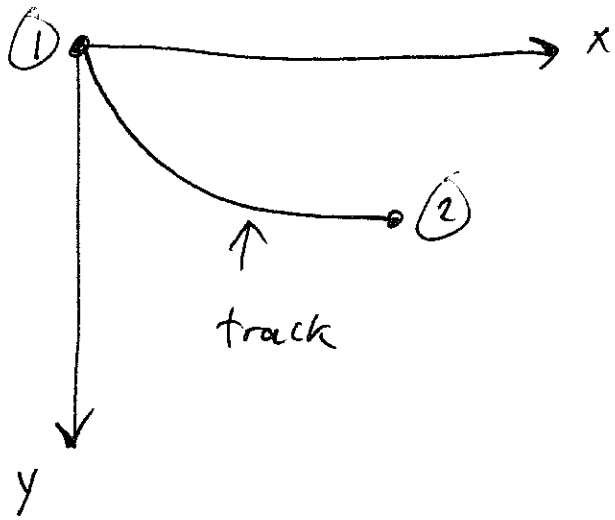
$$(y')^2 = c^2\left(1+(y')^2\right)$$

or $$y' = \frac{\pm c}{\sqrt{1-c^2}} = \text{another constant}$$

$\therefore$ $\boxed{y(x) = mx+b}$, a line

Example: The Brachistochrone. (Famous problem).

In a uniform gravitational field, what shape should we choose for a track in order to minimize the travel time between (1) & (2)?



The time is $\Delta t = \int_1^2 \frac{ds}{v}$

In uniform gravity, $v = \sqrt{2gy}$ by conservation of energy. So let's consider y to be the independent variable, and x to be the dependent variable:

$$x(y).$$

We have $ds = \sqrt{(dx)^2 + (dy)^2}$

$$= \sqrt{(x'(y))^2 + 1}\, dy$$

So $\text{time}(1 \to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{(x'(y))^2 + 1}}{\sqrt{y}}\, dy$

We wish to minimize time. So $x(y)$ must satisfy

$$\frac{\partial F}{\partial x} = \frac{d}{dy}\frac{\partial F}{\partial x'}$$

where $F = \frac{\sqrt{(x')^2+1}}{\sqrt{y}}$

So we have

$$\frac{\partial f}{\partial x} = 0, \text{ and } \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+(x')^2}}$$

So $\frac{x'}{\sqrt{y(1+(x')^2}} = \text{constant} \equiv \frac{1}{2a}$

or $x' = \sqrt{\frac{y}{2a-y}}$

or $x = \int \sqrt{\frac{y}{2a-y}}\, dy$

Substitute $y = a(1-\cos\theta)$ & gives

$$x = a\int (1-\cos\theta)\, d\theta = a(\theta - \sin\theta) + \text{constant}.$$

Since point 1 has $x = y = 0$, the constant should be zero, so

$$x = a(\theta - \sin\theta)$$

and $y = a(1 - \cos\theta)$, where θ = a parameter

a must be chosen so that the path passes through (x_2, y_2). This path is a "cycloid".