Energy Conservation in One-Dimension

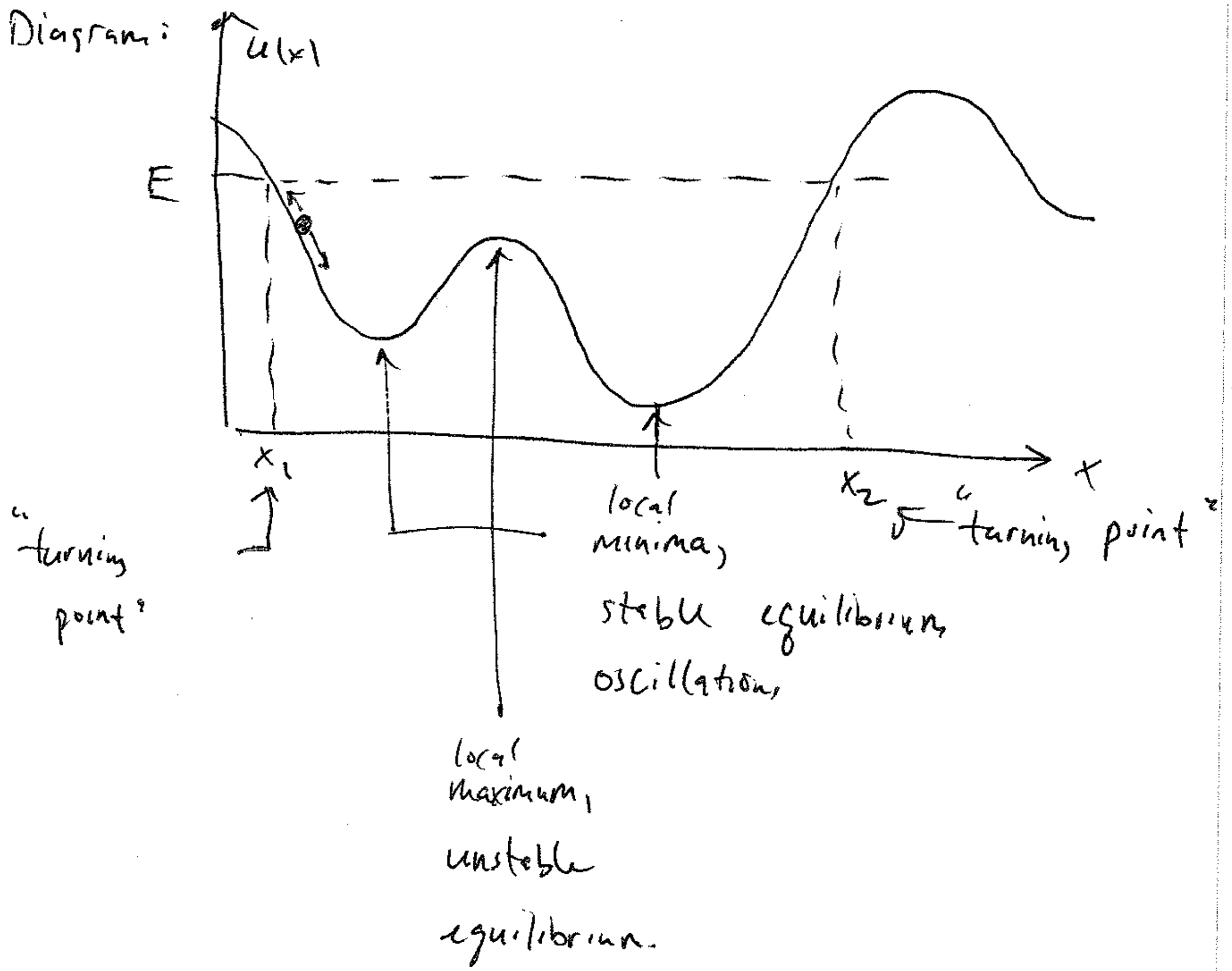
In one dimension (x):

$$W_{12} = \int_{x_1}^{x_2} F(x')\,dx'$$

$$U(x) = -\int_{x_0}^{x} F(x')\,dx'$$

$$F_x = -\frac{dU}{dx}$$

PE Diagram:



In one-dimension conservative systems, it is possible to find a <u>complete solution</u> for the motion from <u>energy considerations alone</u>:

$$T + U = E = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = E$$

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

We can separate varibles and write down an integral to get t as a function of x:

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dx}{\dot{x}} = dt$$

$$\int \frac{dx}{\pm\sqrt{\frac{2}{m}}\sqrt{E-U(x)}} = \int dt = t_f - t_i$$

If we set t_i to zero, we have

$$\boxed{t_f = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x} \frac{dx'}{\sqrt{E-U(x')}}}$$

If we can do this integral, then we can get t as a function of x.

In principle we can then solve for $x(t)$, or, at the very least, solve numerically for any (x) given any (t).

There are some sublties, however:

1) We have to choose whether to use the $(+)$ or $(-)$ sign based on the velocity direction. (Usually we can figure out if v is $(+)$ or $(-)$ from the initial conditions.)

2) If the particle turns around during the motion, then we have to do the integral in pieces, one ~~piece~~ piece for each segment when the velocity is in one direction only.

The most common use of

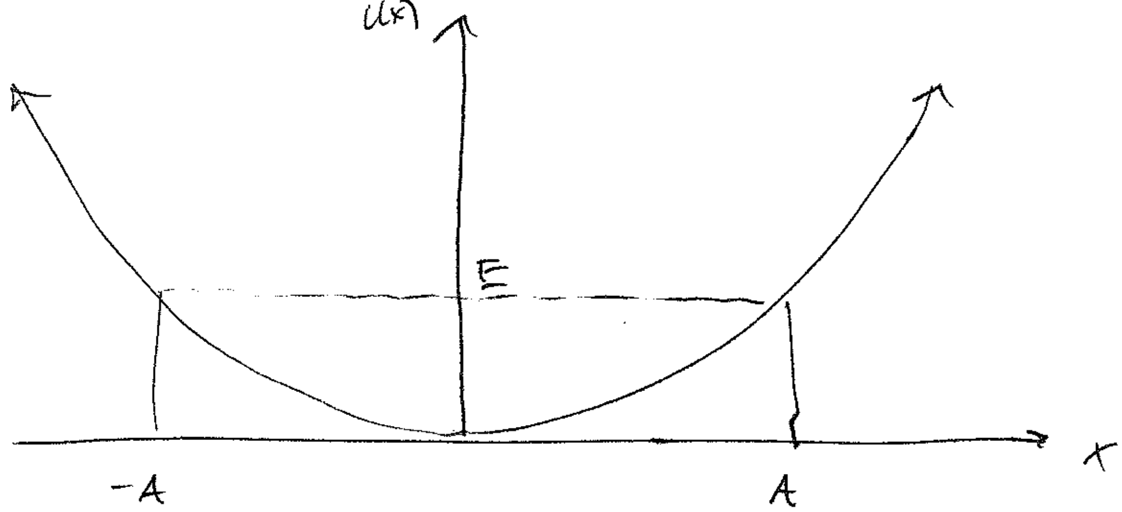
$$t_F = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-U(x')}}$$

is to calculate the <u>period</u> of some oscillatory motion.

<u>Simple Example:</u> <u>Simple Harmonic Motion.</u>

Calculate the period from energy considerations alone.

<u>Solution</u>: $U(x) = \frac{1}{2}kx^2$



Since $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E = \text{constant}$,

$$\dot{x}(x) = \sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}kx^2}$$

This gives the velocity for any position (x).
We can calculate <u>$\frac{1}{4}$ of the period</u> like this

$$t_{\frac{1}{4}} = \sqrt{\frac{m}{2}} \int_{\emptyset}^{A} \frac{dx'}{\sqrt{E - \frac{1}{2}kx'^2}}$$

At the end point, $E = \frac{1}{2}kA^2$, but this is constant so it is always true:

$$t_{\frac{1}{4}} = \underbrace{\frac{\sqrt{m/2}}{\sqrt{\frac{1}{2}k}}}_{\sqrt{\frac{m}{k}} \equiv \frac{1}{\omega_0}} \int_{\emptyset}^{A} \frac{dx'}{\sqrt{A^2 - x'^2}} = \frac{1}{\omega_0}\int_{\emptyset}^{A} \frac{dx'}{\sqrt{A^2 - x'^2}}$$

The integral can be looked up in a table:

$$t_{\frac{1}{4}} = \frac{1}{\omega_0}\left(\sin^{-1}\left(\frac{x'}{A}\right)\Bigg|_{\emptyset}^{A}\right)$$

$$= \frac{1}{\omega_0}\left(\underbrace{\sin^{-1}(1)}_{\pi/2} - \underbrace{\sin^{-1}(\emptyset)}_{\emptyset}\right)$$

$$= \frac{\pi}{2\omega_0}$$

The total period is 4 times larger:

$$\boxed{T = 4t_{\frac{1}{4}} = \frac{2\pi}{\omega_0}}$$

Actually, the entire motion of the system can be found analytically in this case:

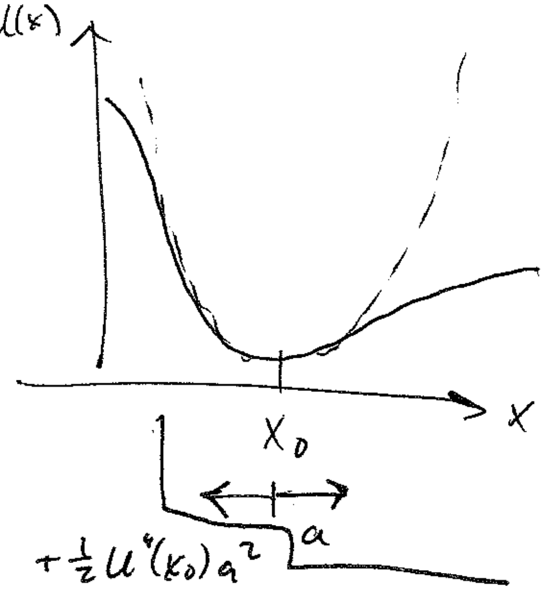
$$t(x) = \frac{1}{\omega_0}\int_{\emptyset}^{x \leftarrow !!!} \frac{dx'}{\sqrt{A^2 - x'^2}}$$

$$= \frac{1}{\omega_0}\left(\sin^{-1}\left(\frac{x}{A}\right) - \underbrace{\sin^{-1}(\emptyset)}_{\emptyset}\right)$$

$$t(x) = \frac{1}{\omega_0}\sin^{-1}\frac{x}{A}$$ ← Solve for $x(t)$:

$$\boxed{x(t) = A\sin(\omega_0 t)}$$

<u>For small oscillations</u>, we can expand the potential near a minimum



Let $x = x_0 + a$

↑ minimum for U

↑ a variable, the distance from the minimum

Then

$$U(x_0+a) \approx U(x_0) + U'(x_0)\,a + \tfrac{1}{2}U''(x_0)\,a^2 + \ldots$$

since U has a minimum at x_0, its slope there is zero: $U'(x_0) = 0$. So

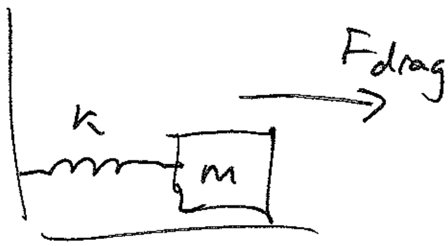
$$U(x_0+a) \approx U(x_0) + \tfrac{1}{2}U''(x_0)\,a^2 + \ldots$$

We see that $U''(x_0)$, the curvature at the minimum, plays the role of the spring constant in a simple harmonic oscillator. So any potential with a non-zero $U''(x_0)$ we execute <u>approximate</u> simple harmonic motion with natural frequency

$$\omega_0 \approx \sqrt{\frac{U''(x_0)}{m}}$$

as long as the amplitude is not too large.

Damped Oscillator, no forcing, linear drag



Letting

$$F_{spring} = -kx \quad \text{and}$$

$$F_{drag} = -bv = -b\dot{x},$$

we have

$$m\ddot{x} + b\dot{x} + kx = 0$$

Or, defining $\frac{b}{m} \equiv 2\beta$, $\frac{k}{m} \equiv \omega_0^2$,

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0} \quad \text{Eq. of Motion}$$

"Homogeneous Equation"

This equation is linear in x, $\dot{x}$, and $\ddot{x}$.

Guess an exponential solution:

$$x(t) = e^{rt}$$. Then substitute:

$$\dot{x} = re^{rt}$$

$$\ddot{x} = r^2e^{rt}$$, so we have

$$r^2 + 2\beta r + \omega_0^2 = 0$$ (sometimes called the auxillary equation)

Algebraic Solution:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \quad \text{or}$$

$$\boxed{\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}}$$

We have 3 cases to consider depending on whether the argument inside the radical is positive, negative, or zero. We can also consider the case of no damping ($\beta = \emptyset$).

<u>No damping</u> ($\beta = \emptyset$): Then $r_1 = +i\omega_0$, $r_2 = -i\omega_0$

$$\boxed{x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}}$$

c_1 & c_2 determined by initial conditions. In fact, c_1 & c_2 are complex, so there are 4 free parameters. But we only want <u>2</u> free parameters for this 2^{nd} order equation so let

$$c_2 = c_1^* \quad . \text{ Then}$$

$$x(t) = c_1 e^{i\omega_0 t} + \underbrace{c_1^* e^{-i\omega_0 t}}$$

complex conjugate of the first term.

But a complex number plus its complex conjugate is twice the real part:

$$z + z^* = 2\,\mathrm{Re}(z)$$

so $x(t) = 2\,\mathrm{Re}\left(C_1 e^{i\omega_0 t}\right)$

$2C_1$ has a magnitude and phase, let them be called A & $-\delta$: $2C_1 = Ae^{-i\delta}$.

Thus

$$x(t) = \mathrm{Re}\left[Ae^{i(\omega_0 t - \delta)}\right]$$

$$\boxed{x(t) = A\cos(\omega_0 t + \delta)}$$

<u>Weak Damping</u>, $\beta < \omega_0$.

Then $\sqrt{\beta^2 - \omega_0^2} = i\underbrace{\sqrt{\omega_0^2 - \beta^2}}_{\text{call this } \omega_1} \equiv i\omega_1$

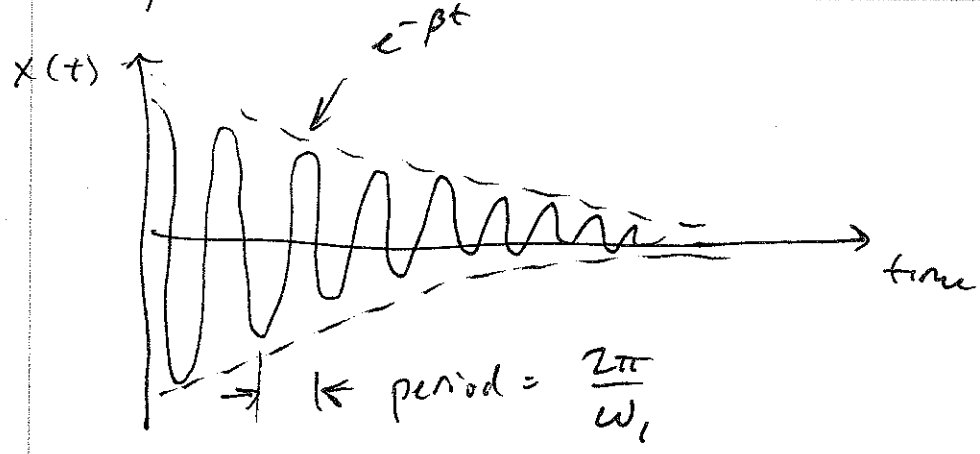
ω_1 is a little bit smaller than ω_0. Then the solution is $(r_1 = -\beta + i\omega_1,\ r_2 = -\beta - i\omega_1)$

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

Again we can combine the terms to get

$$\boxed{x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)}$$

A & δ determined by initial conditions.



Strong Damping (over damped) $(\beta > \omega_0)$

Then $$x(t) = \underbrace{c_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + c_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}}_{\text{real exponential functions}}$$

If you kick an overdamped oscillator at rest, it responds as

Notice that the decay rate is
$$\beta - \sqrt{\beta^2 - \omega_0^2}$$

This rate gets smaller as β gets bigger

meaning that the time to return to equilibrium gets longer as the damping increases.

Critical Damping ($\beta = \omega_0$)

Now our two solutions are degenerate (equal):

$$x(t) = e^{-\beta t}$$

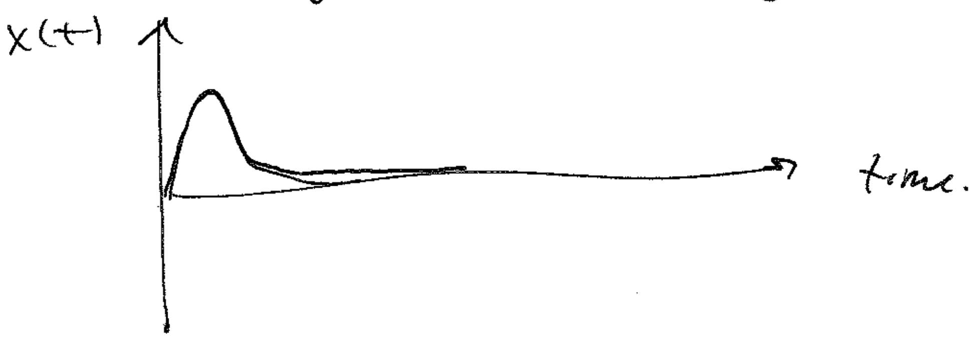
We need a second solution, and this one works:

$$x(t) = t e^{-\beta t}$$

So a general solution is

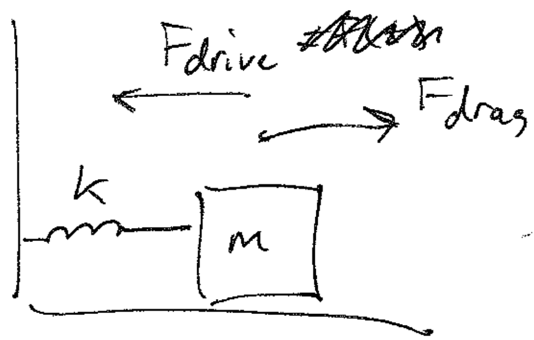
$$x(t) = c_1 e^{-\beta t} + c_2 t e^{-\beta t}$$

A critically damped oscillator returns to equilibrium quickly after being kicked:



Driving Forces: Inhomogeneous Eq. of Motion

Add an external driving force:



Then $$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \mathbf{F}(t)/m \equiv f(t)}$$

driving force, inhomogeneous term.

$$f(t) \equiv \frac{F(t)}{m}$$

We can easily see that there will be two solutions, one of which will have the two free parameters to accomodate initial conditions, and one of which will have zero free parameters.

First define an operator: $D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$

Then our equation of motion is

$$\boxed{Dx = f}$$ ← Inhomogeneous

With no driving force, the Eq. of Motion is

$$\boxed{Dx = 0}$$ ← homogeneous

we'll call the solution to the inhomogeneous Eq the "particular solution" (X_p) and the solution to the homogeneous Eq. the "transient solution" (X_{tr}). ("Transient" solution is also known as the "complementary" solution.)

So we have $DX_p = F$ and $DX_{tr} = \emptyset$

Now X_{tr} has 2 free parameters and has the form

$$X_{tr}(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Now we try $(X_p + X_{tr})$ as a guessed solution to the inhomogeneous Eq:

$$D(X_p + X_{tr}) = DX_p + DX_{tr}$$

because D is a <u>linear</u> operator (this is crucial.)

$$= \underbrace{DX_p}_{F} + \underbrace{DX_{tr}}_{\emptyset}$$

$$= F$$

$\therefore$ $\boxed{D(X_p + X_{tr}) = F}$ So $X_p + X_{tr}$ is a solution to the inhomogeneous Eq.

This is useful because we will see that X_p has zero free parameters. By adding X_{tr} to it, we can accomodate 2 initial conditions.

$X_p \Rightarrow$ no free parameters $\rightarrow$ long-term behavior

$X_{tr} \rightarrow$ 2 free parameters $\rightarrow$ short term behavior

$\rightarrow$ dies away.

Particular Solution, Cosine Driving Force.

Let $f(t) = f_0 \cos(\omega t)$

$\uparrow$ forcing frequency, not ω_0!

Let's solve the Eq. of Motion for the particular solution:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

We use our trick of pretending that this is a complex equation. First rewrite it:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

where $z = x + iy$, or $x = \mathrm{Re}(z)$

Guess a harmonic solution with frequency ω:

$$z = Ce^{i\omega t}$$

Thus $\dot{z} = i\omega z$, $\ddot{z} = -\omega^2 z$

So the Eq. of Motion is

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right)Ce^{i\omega t} = f_0 e^{i\omega t}$$

or $$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$ (an algebraic Eq.)

C is complex, so we can also write it as

$$C \equiv Ae^{-i\delta}$$

Now we can determine A & δ:

$$A^2 = CC^* = \left(\frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}\right)\left(\frac{f_0}{\omega_0^2 - \omega^2 - 2i\beta\omega}\right)$$

$$\boxed{A^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}$$

To get δ:

$$Ae^{-i\delta} = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

$$A\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right) = f_0 e^{i\delta}$$

Taking the phase of both sides, we have

$$\boxed{\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)}$$

Therefore our solution is the real part:

$$x(t) = A\cos(\omega t-\delta)$$

where $A = \dfrac{F_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$ and $\delta = \tan^{-1}\left(\dfrac{2\beta\omega}{\omega_0^2-\omega^2}\right)$

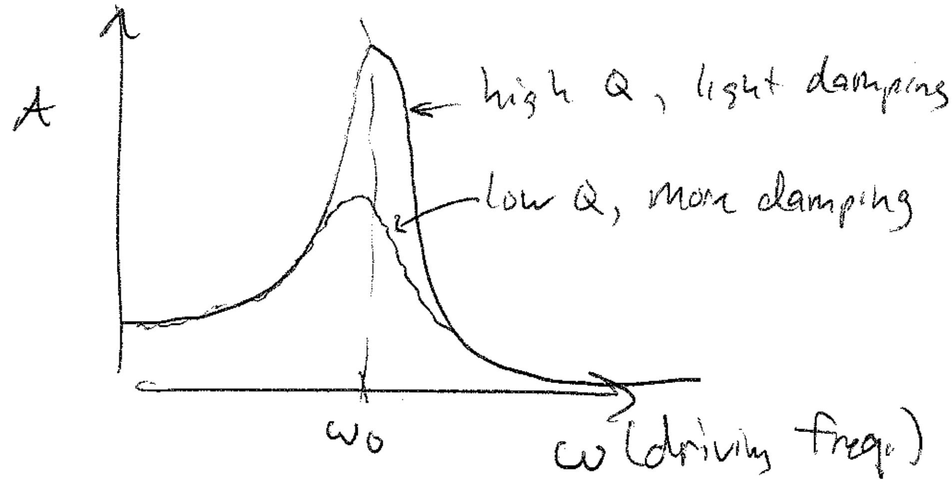
Adding the transient solution we have

$$x(t) = A\cos(\omega t-\delta) + A_{tr}e^{-\beta t}\cos(\omega_1 t-\delta_{tr})$$

damped freq.

2 free parameter

As you know the amplitude of the particular solution (long-term) becomes large if the driving frequency is close to ω_0. This is resonance:



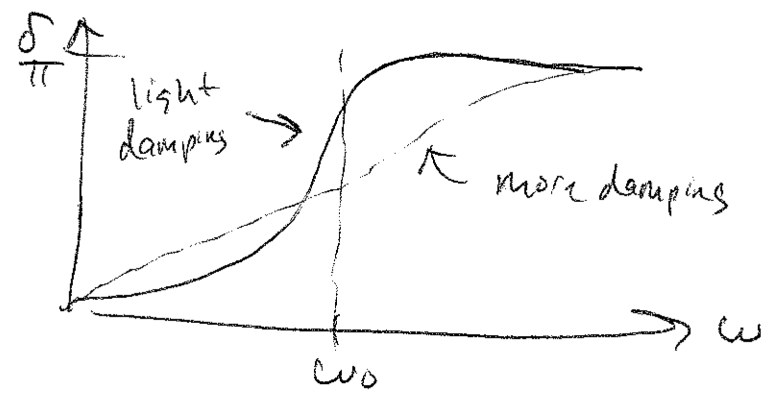
Remember that there are 3 frequencies at play here:

ω_0 = natural frequency of the <u>undamped</u> oscillator

ω_1 = damped frequency, slightly less ~~that~~ than ω_0

ω = driving frequency of the external force.

By the way, the phase shift (δ) looks like



Forced oscillator with <u>any</u> <u>periodic</u> <u>driving</u> <u>force</u>.

A beautiful consequence of the previous analysis is that we can describe <u>any</u> <u>periodic</u> <u>force</u> by using a <u>Fourier Series</u>.

$$F(t) = \sum_{n=0}^{\infty} \left(a_n \cos(n\omega t) + b_n \sin(n\omega t) \right)$$

where $\omega = \frac{2\pi}{T}$, T = The period of the periodic force.

a_n & b_n can be calculated for a particular $F(t)$:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \cos(n\omega t)\, dt, \quad n \geq 1$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} F(t) \sin(n\omega t)\, dt, \quad n \geq 1$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} F(t)\, dt$$

The key point is that because the operator D is linear we can just add together our existing solution for the forced oscillator driven by a single cosine driving force:

Suppose we have two forces and 2 solutions:

$DX_1 = F_1$ and $DX_2 = F_2$. Then

$$D(X_1 + X_2) = DX_1 + DX_2 = F_1 + F_2$$

So the solution for force $F_1 + F_2$ is just $X_1 + X_2$:

If $F(t) = \sum_n f_n(t)$, then

$$x(t) = \sum_n x_n(t) \quad , \text{ where } D x_n = f_n$$

So for a periodic force

$$F(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t)$$

(to keep things simple, assume $F(t)$ is even so there are no sine terms)

Then

$$\boxed{x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)}$$

$$\boxed{\text{where } A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}} \text{ and } \delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)}$$

You can see that any term in the series may be close to resonance, depending on T, ω_0.

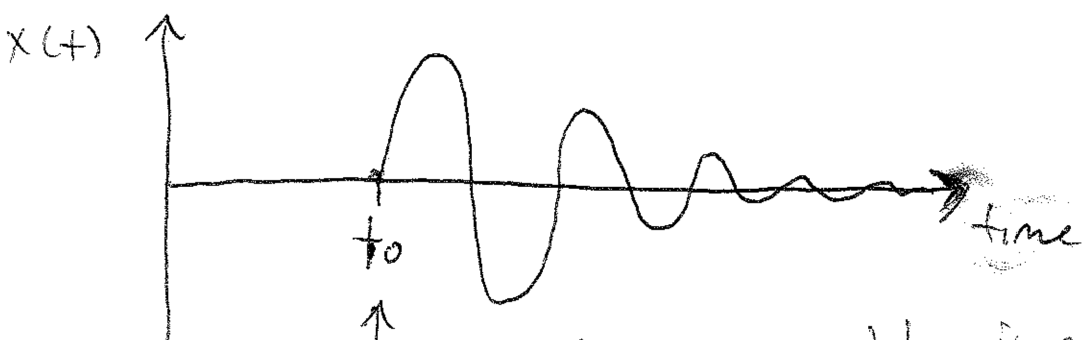
For example, you can expect a significant response amplitude from the oscillator if the period of the force is any multiple of the oscillator's natural period. $\left(\frac{2\pi}{\omega_0}\right)$

Damped Oscillator Solution for an arbitrary (non-periodic) Forcing Function - Green's Method

(This material is from Marion & Thornton, Classical Dynamics, 3rd Edition, section 3.10)

George Green (1793-1841) invented a method for solving linear inhomogeneous differential equation for an arbitrary forcing function, even one which is non-periodic. Green's method is to determine the response of the system to a short sharp blow or impulse, and then to model the force law as a series of such impulses. Since the Eq. of Motion is linear, the solution will be a sum of the response functions.

For a damped oscillator, starting at rest at equilibrium, if we quickly strike the oscillator, the motion will look like:



time at which the blow is applied.

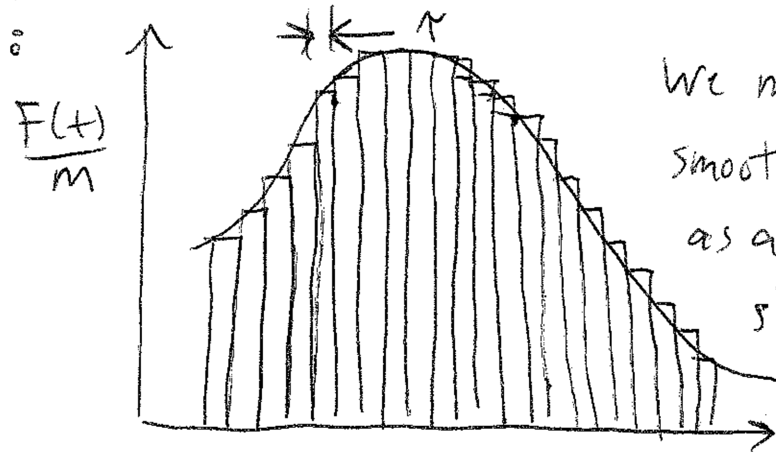
This is the transient solution, and it has the form:

$$x(t) = \begin{cases} \emptyset, & t < t_0 \\ \dfrac{b}{\omega_1} e^{-\beta(t-t_0)} \sin(\omega_1 (t-t_0)), & t > t_0 \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ (damped frequency)

and b = units of m/s = describes the magnitude of the blow.

More precisely, consider a series of short, constant forces:



We model the smooth force function as a series of short, constant forces.

In this approximation, the equation of motion is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \sum_{n=-\infty}^{+\infty} \frac{F_n(t)}{m}$$

Each $\frac{F_n(t)}{m}$ is only non-zero for a short time (τ). During that time it is constant.

$\frac{F_n(t)}{m}$ has units of acceleration, so

let's call it $\frac{F_n(t)}{m} \equiv a_n(t_n)$, $t_n < t < t_{n+1}$

The transient solution for each $\frac{F_n(t)}{m}$ is then

$$x_n(t) = \begin{cases} \frac{a_n(t_n)\tau}{\omega_1} e^{-\beta(t-t_n)} \sin(\omega_1(t-t_n)), & \text{for } t > t_n \\ \emptyset, & \text{for } t < t_n \end{cases}$$

The complete solution is a sum over all (n) up to the N^{th} impulse, which is the most recent impulse:

$$x(t) = \sum_{n=-\infty}^{N} \frac{a_n(t_n)\tau}{\omega_1} e^{-\beta(t-t_n)} \sin(\omega_1(t-t_n)),$$

for $t > t_N$.

As time goes forward, we add more impulse terms to the sum.

Now we recover the continuous force by letting $\tau \to dt'$, $t_n \to t'$. Then

$$x(t) = \int_{-\infty}^{t} \frac{a(t')}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t'))\,dt'$$

We can re-write in terms of the force:

$$F(t') = m\,a(t') \quad \text{by definition of } a(t').$$

So

$$\boxed{x(t) = \int_{-\infty}^{t} F(t') \left[\frac{e^{-\beta(t-t')} \sin\left(\omega_1 (t-t')\right)}{m\omega_1} \right] dt'}$$

This is the solution to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

for an arbitrary forcing function $F(t)$.

The function in brackets is called the "<u>Green's</u> <u>function</u>" for the damped oscillator:

(or more precisely, for the Linear Operator D)

$$D = \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)$$

It is usually written as

$$G(t,t') \equiv \begin{cases} \dfrac{1}{m\omega_1} e^{-\beta(t-t')} \sin\left(\omega_1 (t-t')\right), & \text{for } t \geq t' \\ \emptyset, & \text{for } t < t' \end{cases}$$

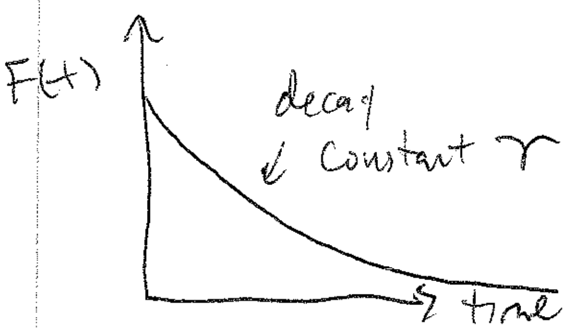
Then $\boxed{x(t) = \int_{-\infty}^{t} F(t')\, G(t, t')\, dt'}$

For any linear inhomogeneous Equation that can be solved by Green's Method, the solution always looks like an integral of the inhomogeneous term ($F(t)$ in our case) times a Green's function. The job is then to determine the correct Green's function. Our Green's function is only appropriate for the damped oscillator equation starting from rest.

<u>Note that the initial conditions are built into the Green's function.</u> We assumed an oscillator starting at rest. A different initial condition would have a different Green's function.

<u>Example solution with Green's Method:</u>

Let $F(t) = F_0 e^{-\gamma t}$, for $t \geq 0$



What is $x(t)$?

Solution:

$$x(t) = \frac{F_0}{m\omega_1} \int_0^t e^{-\gamma t'} e^{-\beta(t-t')} \sin\left(\omega_1 (t-t')\right) dt'$$

Let $z \equiv \omega_1 (t-t')$. Then $dz = -dt' (\omega_1)$

Also $t = \frac{z}{\omega_1} + t'$ or $t' = t - \frac{z}{\omega_1}$

So $t' = 0$ gives $z = \omega_1 t$,

and $t' = t$ gives $z = 0$

and
$$e^{-\gamma t'} e^{-\beta t} e^{\beta t'} = e^{-\beta t} e^{(\beta-\gamma) t'}$$
$$= e^{-\beta t} e^{(\beta-\gamma)\left(t - \frac{z}{\omega_1}\right)}$$
$$= e^{-\gamma t} e^{-(\beta-\gamma) z/\omega_1}$$

$$\therefore \quad x(t) = \frac{-F_0}{m\omega_1^2} \int_{\omega_1 t}^{0} e^{-\gamma t} e^{(\gamma-\beta) z/\omega_1} \sin z \, dz$$

$$= \left(\frac{F_0/m}{(\gamma-\beta)^2 + \omega_1^2}\right) \left[e^{-\gamma t} - e^{-\beta t} \left(\cos\omega_1 t - \frac{\gamma-\beta}{\omega_1} \sin\omega_1 t \right) \right]$$

For $\beta = \gamma$, this looks like: