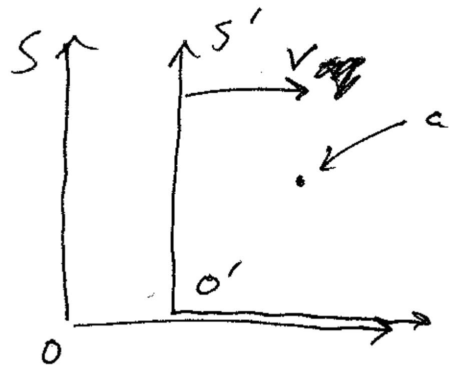
Galilean Relativity:



$x' = x - vt \Rightarrow \dot{x}' = \dot{x} - v$

$y' = y$ and $\ddot{x}' = \ddot{x}$

$z' = z$

$$\Rightarrow \boxed{\begin{array}{l} m\vec{a}' = m\vec{a} \\ \vec{F}' = \vec{F} \end{array}}$$

Newton's 2nd Law appears to have the same form in both systems

However, Maxwell's Eqs appear to violate Galilean Relativity. The problem is that a ~~fundamental~~ velocity appears explicitly → the speed of light. This happens when deriving the wave Eq.

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}, \quad \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \, m/s.$$

In which frame does this wave Eq. hold true with $c = 3 \times 10^8$ m/s? Mechanical waves require a medium, the velocity is measured w/respect to the medium. Suppose that such a medium exists for E&M waves, and suppose an observer travels at the speed of light through that medium.

Then the observer sees a "frozen" EM wave. But such a wave violates basic laws of electrostatics such as

$$\vec{\nabla} \times \vec{E} = \emptyset \quad \left(\text{or } \oint \vec{E} \cdot d\vec{\ell} = \phi\right)$$

because the fixed observer sees

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's Law})$$

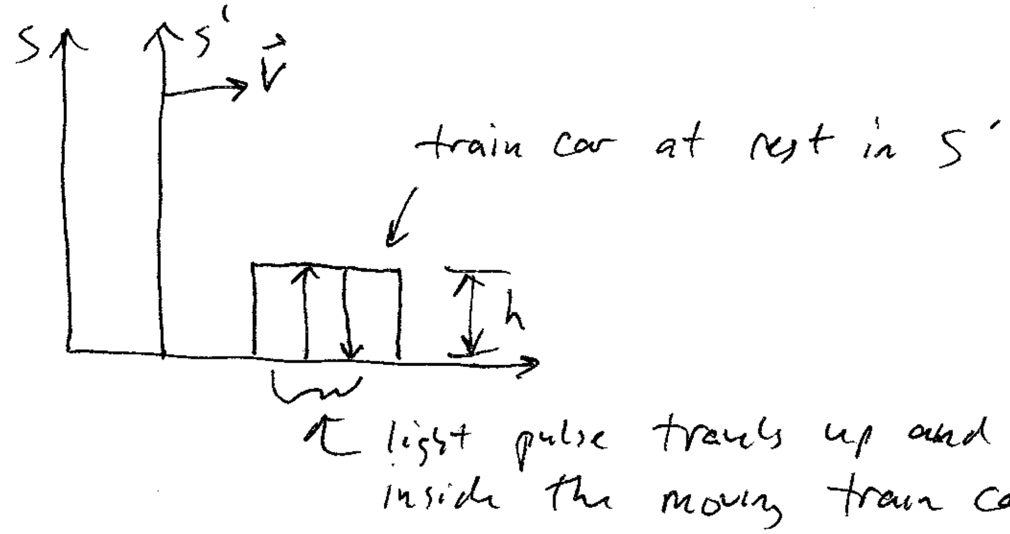
Both observers should observe the $\vec{E}$ field having the same curl ($\vec{\nabla} \times \vec{E}$), because this is just a spatial derivative, which should not change in Galilean Relativity. But the fixed observer sees the magnetic field changing, which satisfies Faradays Law, while the moving observer sees nothing changing, so $\frac{\partial \vec{B}}{\partial t} = \emptyset$, violating Faraday's Law

To fix this, Maxwell's Equations would need to be modified. Einstein and others suggested that we should instead modify our concept of space & time and give up Galilean Relativity. Instead we will modify mechanics to be consistent with the principles of Special Relativity.

(1) Every inertial frame is equivalent $\Rightarrow$ All laws of physics appear to be the same to all inertial observers.

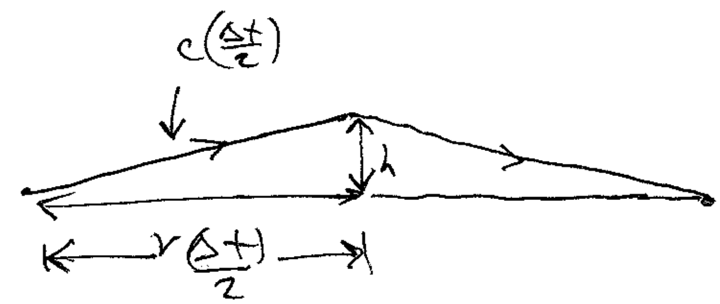
(2) The velocity of light is $c = 3\times10^8$ m/s observed in all inertial frames (violating Galilean Relativity where ~~[illegible]~~ $\dot{x}' = \dot{x} \pm V$.)

Time Dilation



In S': $\Delta t' = \frac{2h}{c}$

In S:



$$\left(c\left(\frac{\Delta t}{2}\right)\right)^2 = \left(v\frac{(\Delta t)}{2}\right)^2 + h^2$$

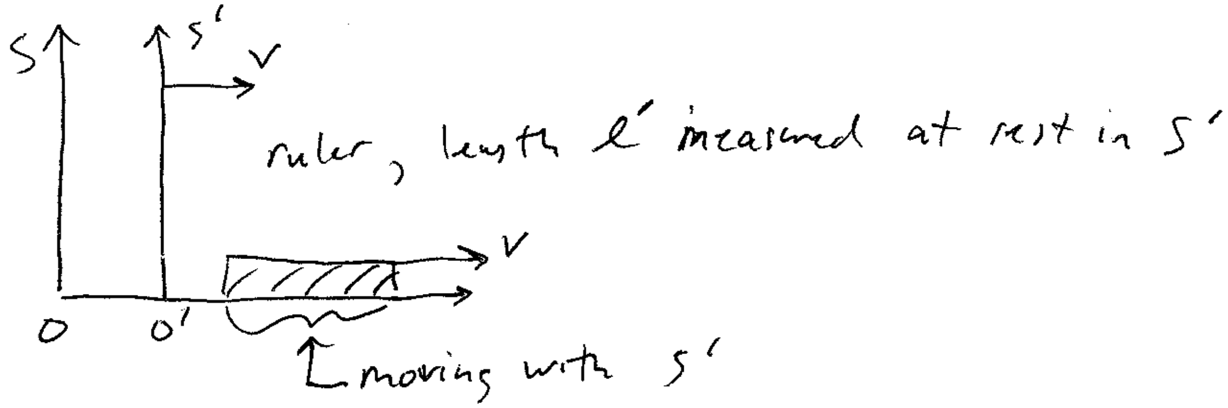
or $\Delta t = \dfrac{2h}{\sqrt{c^2-v^2}} = \underbrace{\dfrac{2h}{c}}_{\Delta t'} \dfrac{1}{\sqrt{1-\beta^2}}$, $\boxed{\beta \equiv \dfrac{v}{c}}$

or $\boxed{\Delta t = \dfrac{\Delta t'}{\sqrt{1-\beta^2}}} \equiv \boxed{\gamma(\Delta t')}$, where $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}}$

So the two observers measure different times for the light to travel up and back.
Two events observed at the <u>same location in space</u> have ~~in the rest frame (where the~~ light time difference ($\Delta t'$).
we call this the <u>proper time</u>.

<u>Length Contraction</u>

A ruler is moving in frame S, but at rest in S':



How long is the ruler as seen in S?
How long does it take the ruler to pass by a location fixed in S (such as the origin)?
We can multiply that time by the velocity v to get the ruler's length in S.

In S', an observer sees the ruler at rest with length ℓ', but also sees a point fixed in S as traveling with speed $|v|$.

$$\ell' = |v|(\Delta t')$$

← the time that a point fixed in S takes to pass the full length of the ruler, as observed in S'.

In S, an observer watches the ruler pass by a fixed location, taking time Δt:

$$\ell = |v|(\Delta t)$$

length in S ↑ ← time for ruler to pass by in S.

Since the observer in S sees the two events at the same location in space. So Δt is the proper time in this case. Then

$$\Delta t' = \gamma(\Delta t), \quad \text{so} \quad \ell' = |v|\gamma(\Delta t)$$

$$\ell' = \gamma \underbrace{|v|(\Delta t)}_{\ell}$$

$$\ell' = \gamma \ell$$

or $$\boxed{l = \frac{l'}{\gamma} \leq l'}$$

The frame S' is special in this experiment, because the ruler is at rest there we call the length of the ruler observed at rest the "proper length", and use symbol l_0

so
$$\boxed{l = \frac{l_0}{\gamma} \leq l_0}$$

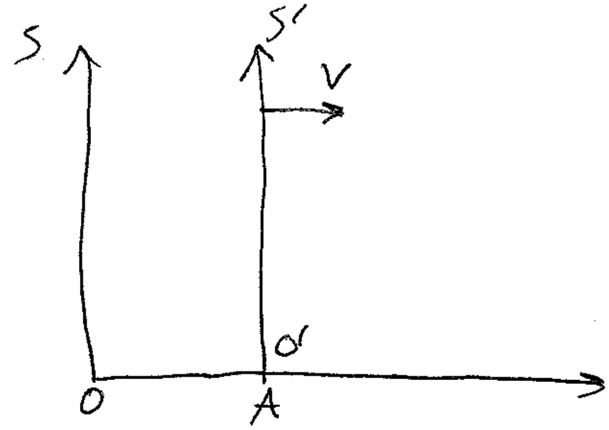
The length observed when the ruler is in motion is shorter by factor $\frac{1}{\gamma}$

(note that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is ≥ 1).

Lorentz Transformation

Let ΔX and Δt be the spatial difference and time difference between 2 events measured in S. (And let $\Delta X'$ and $\Delta t'$ be the quantities measured in S'.) We wish to determine a transformation matrix

$$\begin{pmatrix} \Delta X' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \Delta X \\ c\Delta t \end{pmatrix}$$



Event 1: O' and O' coincide

Event 2: O' and A coincide

proper time

In S: $\Delta x = V\Delta t$

In S': $\Delta x' = \emptyset$, $\Delta t' = (\Delta t)\sqrt{1-\beta^2} = \dfrac{\Delta t}{\gamma}$

both happen at O', which is fixed in S'

Our transformation reads as

$$\Delta x' = a_1 \Delta x + a_2 (c\Delta t)$$

$$\Downarrow$$

$$\emptyset = a_1 \Delta x + a_2 (c\Delta t)$$

$$\Longrightarrow \quad \frac{a_2}{a_1} = -\frac{\Delta x}{c\Delta t} = -\frac{V}{c} = -\beta$$

For the transformation of $c\Delta t'$, we have

$$c(\Delta t') = a_3 \Delta x + a_4 c(\Delta t)$$

$$c(\Delta t') = a_3 (V\Delta t) + a_4 (c\Delta t)$$

$$c(\underbrace{\Delta t'}_{(\Delta t)\sqrt{1-\beta^2}}) = (a_3 V + a_4 c)\Delta t$$

$$c\sqrt{1-\beta^2} = a_3 V + a_4 c$$

$$\sqrt{1-\beta^2} = a_3\beta + a_4$$

So our transformation appears as

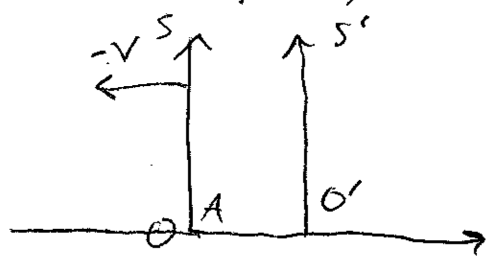
$$\begin{pmatrix} \emptyset \\ \sqrt{1-\beta^2} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Now reverse our point of view. We now see S' as fixed, and S travels backwards at speed $-V$ along $-\hat{x}$. Then we can repeat the above scenario, with everything the same except

$\Delta x \leftrightarrow \Delta x'$

$\Delta t \leftrightarrow \Delta t'$

$V \leftrightarrow -V$.



Then in S', $\Delta x' = -V\Delta t'$

in S, $\Delta x = \emptyset$, while $\underset{\uparrow \text{ proper time}}{\Delta t} = \Delta t'\sqrt{1-\beta^2}$

The transformation appears as

$$\begin{pmatrix} -V\Delta t' \\ c\,\Delta t' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ c\Delta t'\sqrt{1-\beta^2} \end{pmatrix}$$

Divide everything by $c\Delta t'$:

$$\begin{pmatrix} -\beta \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{1-\beta^2} \end{pmatrix}$$

Now we can read off the following:

$$-\beta = a_2\sqrt{1-\beta^2} \implies \boxed{a_2 = \frac{-\beta}{\sqrt{1-\beta^2}}}$$

And we already know $\frac{a_2}{a_1} = -\beta$, so we must have

$$\boxed{a_1 = \frac{1}{\sqrt{1-\beta^2}}}$$

Similarly we can read off

$$1 = a_4\sqrt{1-\beta^2} \implies \boxed{a_4 = \frac{1}{\sqrt{1-\beta^2}}}$$

From before we have

$$\sqrt{1-\beta^2} = a_3\beta + a_4 = a_3\beta + \frac{1}{\sqrt{1-\beta^2}}$$

This is solved by $\boxed{a_3 = \frac{-\beta}{\sqrt{1-\beta^2}}}$

Finally the Lorentz Transformation is

$$\boxed{\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}}$$

or

$$\boxed{\begin{aligned} \Delta x' &= \gamma(\Delta x - v\Delta t) = \frac{\Delta x - v\Delta t}{\sqrt{1-\beta^2}} \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1-\beta^2}} \end{aligned}}$$

The inverse transformation replaces primed variables by unprimed variables and v by $-v$:

$$\boxed{\begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix}}$$

Sometimes we use the notation $\eta \equiv \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}}$

Then the transformation matrix is

$$\begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix}$$

Also notice that $\gamma^2 - \eta^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = 1$

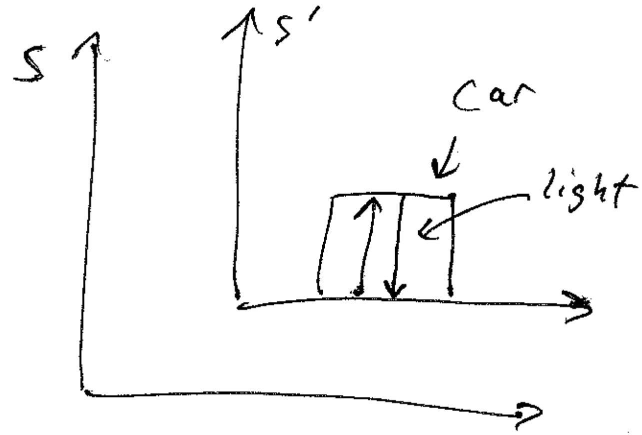
$$\boxed{\gamma^2 - \eta^2 = 1}$$

Now that we have the Lorentz Transformation it is easy to re-derive time dilation and length contraction:

Time Dilation from the Lorentz Transformation

In S',
$\Delta x'$ between light beam leaving and returning is zero: $\Delta x' = \phi$

$\Delta t'$ = nonzero.



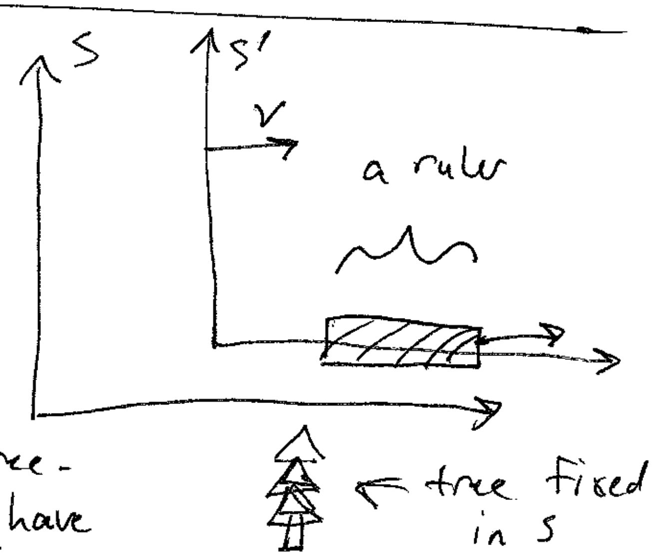
What is Δt in S? Use the Lorentz Transformation

$$\begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \eta \\ \eta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x' = \phi \\ c\Delta t' \end{pmatrix} \Rightarrow \boxed{\Delta t = \gamma \underbrace{\Delta t'}_{\text{proper time}}} > \Delta t'$$

dilated time

Length Contraction from Lorentz Transformation.

In S, the length is $\ell = v(\Delta t)$, where Δt is time between front and back of the ruler being next to the fixed tree. These two spacetime events have $\Delta x = \phi$.



In S', the length is ℓ', which is the proper length (because the ruler is fixed in S').

We can calculate ℓ' in terms of the Δx and Δt:

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \emptyset \\ c\Delta t \end{pmatrix}$$

$$\Rightarrow \Delta x' = \text{proper length} \equiv \ell' = -\eta c \Delta t$$
$$= -\underbrace{(\beta c)}_{v}\gamma \Delta t$$
$$= -\underbrace{(v)(\Delta t)}_{\ell}\gamma$$
$$= -\ell\gamma$$

$$\Rightarrow \boxed{\text{proper length} \left|\frac{(\ell')}{\ell}\right| = \gamma} > 1$$

$$\Rightarrow \ell < \ell'$$

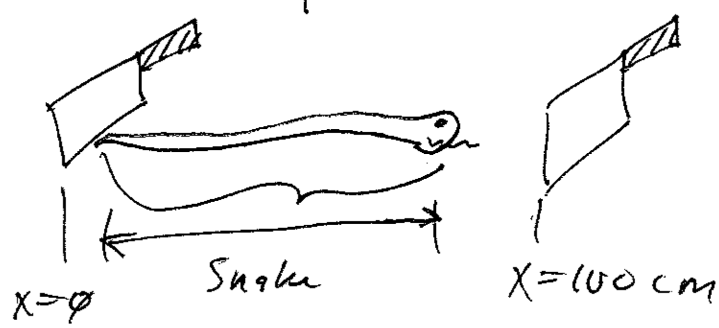
Simultaneous Events

Because of time dilation, observers in different frame cannot all agree on what events occur simultaneously. $\Rightarrow$ time is relative (depends on your frame of reference.)

Snake paradox (conundrum)

A lab has 2 cleavers set 100 cm apart.
~~A 140 cm snake~~
A snake whose proper length is 100 cm travels between the cleavers at velocity $v = 0.6\,c$ ($\beta = 0.6$). ~~The~~ In the lab frame the cleavers come down when the snake's tail first clears the left cleaver:

Lab Frame point of view:

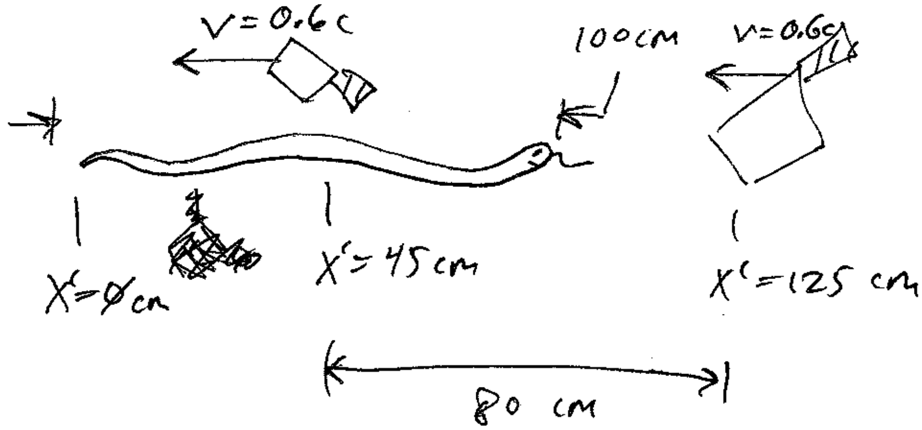


Snake appears 80 cm long in lab frame, so it is safe.

However, isn't it true that in the snake's frame, the cleavers appear only 80 cm apart, while the snake is 100 cm long? Answer: yes, this is how the snake sees the situation. So does the snake get cut from its point of view?

Answer: No. From the snake's point of view, the two cleavers do not come down at the same time:

At $t' = \underline{-2.5 \text{ ns}}$, the snake sees the right cleaver come down at $x' = 125$ cm:



At $t' = 0$, the left cleaver comes down.
$\Rightarrow$ If the right cleaver does not rise, then the snake is hit on the head by the blunt side of the right cleaver!
(But this happens in both frames.)

We can calculate this:

Frame S (cleaver frame):

length of snake in cleaver frame $= \dfrac{100 \text{ cm}}{\gamma} = 80 \text{ cm}$, $\quad \gamma = \dfrac{1}{\sqrt{1-(0.6)^2}} = 1.25$

Δt = time between chops $= 0$

In the snake frame, the right cleaver comes down at

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} 100\text{ cm} \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{pmatrix} \begin{pmatrix} 100 \\ \emptyset \end{pmatrix}$$

$$\boxed{\Delta x' = 125\text{ cm}}$$

$$c\Delta t' = -75\text{ cm} \implies \boxed{\Delta t' = -2.5\text{ nano seconds}}$$

Some Formalism

Curious Fact: the quantity

$$(\Delta s)^2 \equiv (\Delta x)^2 - (c\Delta t)^2$$

has the same value in any frame of reference. We call Δs the "space-time interval" and we say that it is a "Lorentz Invariant"

Proof:

$$(\Delta x)^2 - (c\Delta t)^2 = (\gamma \Delta x' + \gamma\beta c\Delta t')^2 - (\gamma\beta\Delta x' + \gamma c\Delta t')^2$$

$$= \gamma^2(\Delta x' + \beta c\Delta t')^2 - \gamma^2(\beta\Delta x' - c\Delta t')^2$$

(cross terms now cancel)

$$= \underbrace{\gamma^2(1-\beta^2)}_{1}(\Delta x')^2 + \underbrace{\gamma^2(\beta^2-1)}_{-1}(c\Delta t')^2$$

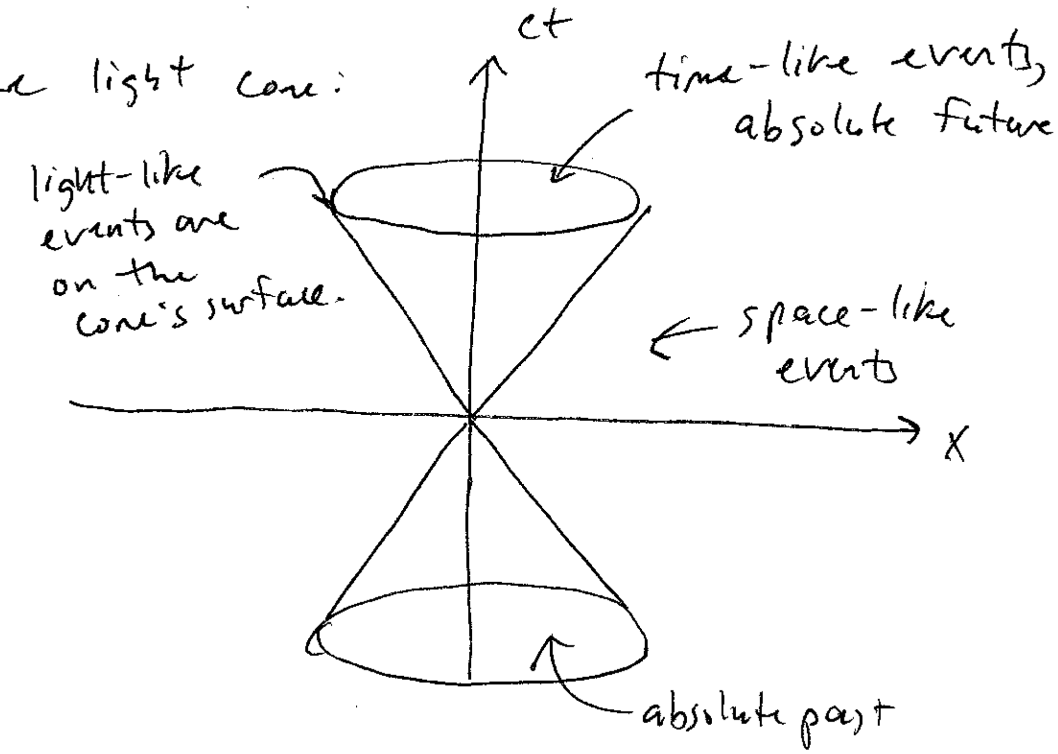
$$\boxed{= (\Delta x')^2 - (c\Delta t')^2}$$

So all observers agree on the numerical value of any (spacetime interval)^2

We categorize spacetime intervals as follows:

- $(\Delta s)^2 > \emptyset \Rightarrow$ "space-like" $\Rightarrow$ the two events are causally disconnected. Their order can be reversed by going to another frame of reference.
- $(\Delta s)^2 = \emptyset \Rightarrow$ "light-like"
- $(\Delta s)^2 < \emptyset \rightarrow$ "time-like" $\Rightarrow$ the two events are causally related. Their order cannot be reversed by changing frames.

The light cone:



Formalism

We notice that the space-time interval Δs is calculated in a way that is similar to a dot product of a vector with itself:

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2$$

The only difference is that we use a $(-)$ sign for the ~~Δx~~ $(c\Delta t)^2$ part rather than a $(+)$ sign. In fact, if we include y & z, we have

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

Δy and Δz are the same in all reference frames, so $(\Delta s)^2$ is still a Lorentz Invariant.

We now define a new type of dot product that puts the $(-)$ sign in the correct place.

Let $\vec{X} = (x, y, z, ct)$

We want to have a dot product like this:

$$(\Delta s)^2 \equiv \vec{X} \cdot \vec{X} = x^2 + y^2 + z^2 - (c\Delta t)^2$$

So let's have 2 types of $\vec{x}$ vectors:

$$x = x_\mu = (x, y, z, ct) \leftarrow \text{"covariant 4-vector"}$$

$\mu = 1, 2, 3, 4$

$$x = x^\mu = (x, y, z, -ct) \leftarrow \text{"contravariant 4-vector"}$$

(-) sign!

Notice that when μ is a superscript (x^μ), the the vector has the (-) sign on the time component. Also notice the μ is a vector index running from 1 to 4:

$$\begin{aligned} x_1 &= x & x^1 &= x \\ x_2 &= y & x^2 &= y \\ x_3 &= z & x^3 &= z \\ x_4 &= ct & x^4 &= -ct \end{aligned}$$

To take the dot product of 4-vector x with itself to calculate a space-time interval, we must always multiply x_μ by x^μ and sum over μ:

$$(\Delta s)^2 = \sum_{\mu=1}^{4} x_\mu x^\mu = xx + yy + zz + (ct)(-ct)$$
$$= x^2 + y^2 + z^2 - (ct)^2$$

we must always have one index upstairs (superscript) and one index downstairs (subscript) to take the dot product.

We now define a 2nd rank tensor (matrix) which changes a covariant vector to a contravariant vector:

$$g^{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{"metric tensor"}$$

Now we can convert between x_ν and x^μ:

$$x^\mu = \sum_{\nu=1}^{4} g^{\mu\nu} x_\nu$$

This means

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

or

$$x^1 = x_1$$
$$x^2 = x_2$$
$$x^3 = x_3$$
$$x^4 = -x_4 \quad \text{as desired.}$$

Similarly, we define

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

With $g_{\mu\nu}$ we can do

$$x_\mu = \sum_{\nu=1}^{4} g_{\mu\nu} x^\nu$$

Notice that our notation becomes very clean if we use <u>Einstein summation notation</u>:

> Any repeated index, with one a superscript and one a subscript, implies a sum from 1 to 4.

Then $\quad x_\mu = \overbrace{g_{\mu\nu} x^\nu}^{\text{sum}}$

and $\quad x^\mu = \underbrace{g^{\mu\nu} x_\nu}_{\text{sum}}$

Then $\quad (\Delta s)^2 = x \cdot x = \underbrace{x_\mu x^\mu}_{\text{sum}} = x_\mu \left(g^{\mu\nu} x_\nu \right)$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$

$$= (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$$

as desired.

Furthermore, let's define

$$g_\mu{}^\nu = g^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

one index up, and one index down!

(+) sign!

With this notation convention, moving any index from upstairs to downstairs or vice-versa has the effect of reversing the sign of the 4th component.

note that, by definition,

$$g_\mu{}^\nu = g^\mu{}_\nu = \delta_{\mu\nu}$$

Kronecker Delta.

4-vectors and the invariance of the scalar product

We will approach special relativity by re-writing all the familiar laws of Newtonian Mechanics in terms of 4-vectors such as (x, y, z, ct). If a law is written in terms of 4-vectors, then it explicitly complies with the requirements of special relativity, because 4-vectors transform in the correct way under a change of frame of reference.

We can determine a requirement on $g_{\mu\nu}$ and the Lorentz Transformation matrix by requiring that the scalar product of any 2 4-vectors is the same in any frame of reference. To see this relationship, let

A & B be 4-vectors, as measured in frame S.

In Frame S', A & B are called A' & B'. The Lorentz Transformation Matrix tells us how A' is related to A and how B' is related to B.

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\eta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

and similarly for B' & B. Now let the Lorentz Transformation Matrix be called Λ (capital lambda). Then

$A' = \Lambda A$ is the transformation of A

and $B' = \Lambda B$ is the transformation of B.

In summation notation:

$$A'^{\nu} = \underbrace{\Lambda^{\nu}{}_{\mu} A^{\mu}}_{\text{summation implied}} \quad \text{and} \quad B'^{\alpha} = \underbrace{\Lambda^{\alpha}{}_{\beta} B^{\beta}}_{\text{summation implied}}$$

Note that $\Lambda^{\nu}{}_{\mu}$ and $\Lambda^{\alpha}{}_{\beta}$ represent the <u>same matrix</u>. We give them different dummy indices because the implied sums are independent of each other.

The scalar product in S' is:

$$A' \cdot B' = \underbrace{A'_{\alpha} B'^{\alpha}}_{\text{summation implied}} = \underbrace{\left(g_{\alpha\nu} A'^{\nu}\right)}_{A'_{\alpha}} B'^{\alpha}$$

$$= g_{\alpha\nu} \underbrace{\left(\Lambda^{\nu}{}_{\mu} A^{\mu}\right)}_{A'^{\nu}} \underbrace{\left(\Lambda^{\alpha}{}_{\beta} B^{\beta}\right)}_{B'^{\alpha}}$$

Now we require that this scalar product be the same when calculated directly in frame S:

In S: $A \cdot B = A_{\beta} B^{\beta} = g_{\beta\mu} A^{\mu} B^{\beta}$

So we demand that

$$g_{\alpha\nu} \Lambda^{\nu}{}_{\mu} A^{\mu} \Lambda^{\alpha}{}_{\beta} B^{\beta} = g_{\beta\mu} A^{\mu} B^{\beta}$$

or

$$\left(\Lambda^{\alpha}{}_{\beta}\, g_{\alpha\nu}\, \Lambda^{\nu}{}_{\mu}\right)\underbrace{\left(A^{\mu} B^{\beta}\right)} = g_{\beta\mu}\underbrace{\left(A^{\mu} B^{\beta}\right)}$$

same

$$\therefore \quad \boxed{\Lambda^{\alpha}{}_{\beta}\, g_{\alpha\nu}\, \Lambda^{\nu}{}_{\mu} = g_{\beta\mu}}$$

In Matrix Notation, this reads as

$$\begin{pmatrix} \gamma & 0 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\eta & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\eta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

or

$$\boxed{\Lambda^{T} G \Lambda = G}$$

Note that this follows from $\gamma^2 - \eta^2 = 1$

when G is the matrix form of $g_{\mu\nu}$.

This equation says that the metric tensor $g_{\mu\nu}$ is unchanged under a Lorentz Transformation.

In fact, a better definition at of the Lorentz Transformation group is that it is the set of all matrices Λ that leave $g_{\mu\nu}$ unchanged

Relativistic velocity Addition

In frame S, $\vec{v} = \frac{d\vec{r}}{dt}$

In frame S',
$$dx' = \gamma(dx - V\,dt)$$
$$dy' = dy$$
$$dz' = dz$$
$$dt' = \gamma(dt - V\,dx/c^2)$$

so
$$v_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - V\,dt)}{\gamma(dt - V\,dx/c^2)}$$

or
$$\boxed{v_x' = \frac{v_x - V}{1 - v_x V/c^2}}$$

Also
$$v_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - V\,dx/c^2)}$$

$$\boxed{v_y' = \frac{v_y}{\gamma(1 - v_x V/c^2)}}, \quad \boxed{v_z' = \frac{v_z}{\gamma(1 - v_x V/c^2)}}$$

where $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$

Example: A bullet is fired at speed 0.6c from a rocket traveling at 0.8c. Relative to the earth, how fast does the bullet travel?

Answer: Our usual convention is that S is fixed and S' is moving. We can infer that

$$v = \frac{v' + V}{1 + v'V/c^2}$$

by reversing the sign of V and exchanging $v' \leftrightarrow v$.

Then $$v = \frac{0.6c + 0.8c}{1 + (0.8)(0.6)} = \frac{1.4c}{1.48} = 0.95c.$$

So the velocity is less than c as measured by earth.

<u>4-velocity</u>

The three-velocity is $\vec{v} = \frac{d\vec{x}}{dt}$,

where dt is the time as seen by a fixed observer. We can construct a 4-vector for the velocity by considering a related quantity

$$\vec{u} \equiv \frac{d\vec{x}}{dt_0},$$ where dt_0 is the <u>proper time</u>. Since the proper time a Lorentz Invariant, the vector $\vec{u}$ is useful for defining a 4-vector for velocity.

We consider the 4-velocity to be

$$u = \frac{dx}{dt_0} = \left(\frac{d\vec{x}}{dt_0}, c\frac{dt}{dt_0}\right)$$

$$\boxed{u = \left(\gamma \frac{d\vec{x}}{dt}, \gamma c\right) = \text{4-velocity}}$$

where $\gamma = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}$

Notice that the ordinary 3-velocity is not simply the first three components of the 4-velocity. Instead the 3-velocity is the first 3 components divided by γ, $\left(\gamma = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}}\right)$

The 4-velocity is useful primarily for defining the 4-momentum:

$$\boxed{p \equiv mu = (\gamma m\vec{v}, \gamma mc) = \text{4-momentum}}$$

Since m is a lorentz scalar (Lorentz Invariant), and u is a 4-vector, this means that p is a Lorentz 4-vector.

~~~~ also

We define the energy such that the 4th component of the 4 momentum is $E/c$:

$$E/c \equiv p_4 = \gamma mc$$

or $\boxed{E = \gamma mc^2}$
~~~~

Then the 4-momentum is

$$p = (\vec{p}, E/c)$$

where $\vec{p} = \gamma m \vec{v}$.

Does our definition of E make sense? Let's expand

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \ldots\ldots$$

Then
$$E = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\ldots\right)(mc^2)$$

$$= mc^2 + \frac{1}{2}mv^2 + \ldots\ldots$$

where mc^2 is the rest-mass energy and $\frac{1}{2}mv^2 + \ldots\ldots$ is the kinetic energy.

The important feature of this result is that in some reactions, such as particle decays the rest mass energy can change. This is fine as long as the relativistic energy E is conserved. So kinetic energy can become rest-mass energy and vice-versa.

Apparently we have $E = mc^2 + T$, $T \to$ kinetic energy.

or $$\boxed{T = E - mc^2 = (\gamma - 1)mc^2}$$

The kinetic energy T is only a useful quantity if you know that m has not changed. In this case T must be conserved so that E is conserved.

Summarizing, we have two new 4-vectors.

4-velocity : $u = (\gamma\vec{v}, \gamma c)$

4-momentum : $p = (\vec{p}, E/c)$

where $\vec{p} = \gamma m \vec{v}$.

Since these quantities are 4-vectors, they transform according to the Lorentz Transformation:

$$\begin{pmatrix} \gamma v_x' \\ \gamma c \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} \gamma v_x \\ \gamma c \end{pmatrix} \Bigg\} \text{4-velocity transformation}$$

and

$$\begin{pmatrix} p_x' \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\eta \\ -\eta & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ E/c \end{pmatrix} \Bigg\} \text{4-momentum transformation}$$

Invariant mass

Note that the length of the 4-momentum is simply the negative of the invariant mass. We can see this by considering the

4-momentum as it appears in the particle's rest-frame. In this frame,

$$p = (\emptyset, \emptyset, \emptyset, mc).$$

So $\quad p \cdot p = (\emptyset)(\emptyset) - (mc)^2 = -(mc)^2$

But since p is a 4-vector, its length must be the same in all reference frames. So the length is $-(mc)^2$ always.

We can go a step further by noting that in any other frame where the particle is moving we have

$$p = (\vec{p}, E/c)$$

$$p \cdot p = \vec{p}^{\,2} - \left(\frac{E}{c}\right)^2$$

And this must be equal to $-(mc)^2$:

$$\vec{p}^{\,2} - \left(\frac{E}{c}\right)^2 = -(mc)^2$$

or $\boxed{E^2 = (pc)^2 + (mc^2)^2}$ This is the most useful relation in all of relativistic kinematics.

We usually try to avoid thinking about velocity, and instead think about E, p, and m, which are related by this formula.

In those instances where we want to know the velocity, we calculate it like this:

$$\frac{pc}{E} = \frac{(\gamma m v)c}{\gamma m c^2} = \frac{v}{c} = \beta$$

or $$\boxed{\beta = \frac{pc}{E}}$$

This is much better than trying to calculate v or β by solving for β in $\frac{1}{1-\beta^2}$, because for high energy particles ~~the~~ a typical calculator will not be accurate enough.

Note that if we choose our system of units such that the speed of light $= 1$, then our kinematic formulas are very simple:

$$E^2 = p^2 + m^2, \quad \beta = \frac{p}{E}$$

$$p = (\vec{p}, E), \text{ and } u = (\gamma\vec{\beta}, \gamma)$$

$$\text{or } u = (\vec{\eta}, \gamma)$$

$$\text{where } \vec{\beta} = \left(\frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right)$$

$$\text{and } \vec{\eta} = \vec{\beta}\gamma$$

Summary of Relativistic Kinematics.

1) Always use $\boxed{E^2 = (pc)^2 + (mc^2)^2}$

or $E^2 = p^2 + m^2$ (with $c = 1$)

2) Avoid using $E = \gamma mc^2$

or $E = \gamma m$

unless you already know γ.

3) To get the velocity use

$$\beta = \frac{pc}{E}$$

or $$\beta = \frac{p}{E}$$

4) Avoid getting β from γ.

~~Example~~

5) Energy-momentum conservation means

$$\underbrace{P_{initial} = P_{Final}}_{\text{4-vectors}}$$

or $(p_x, p_y, p_z, E/c)_{initial} = (p_x, p_y, p_z, E/c)_{final}$

c) From energy-momentum conservation, it must be true that

$$\underbrace{|P_{\text{initial}}|^2}_{} = \underbrace{|P_{\text{Final}}|^2}_{}$$

Invariant mass is the same before and after

But the invariant mass is <u>not</u> just the simple sum of the masses of the particles.

<u>Ex</u>: $\pi^+ \rightarrow \mu^+ \nu_\mu$

π^+: $M = 140$ MeV; μ^+: $M = 106$ MeV; ν_μ: $M = 0$

So it's <u>not true</u> that $M_\pi = M_\mu + M_\nu$

To see why, look at the final state 4-momentum:

$$P_{\text{Final}} = (0, 0, 0, (E_\mu + E_\nu)/c)$$

$$\left(|M_{\text{invariant}}|c\right)^2 = -\frac{(E_\mu + E_\nu)^2}{c^2}$$

$$= -\frac{(E_\mu^2 + E_\nu^2 + 2E_\mu E_\nu)}{c^2}$$

$$= -\frac{\left((m_\mu c^2)^2 + (p_\mu c)^2 + (p_\nu c)^2 + 2E_\mu E_\nu\right)}{c^2}$$

The μ^+ momentum and ν momentum contribute to the final state invariant mass. It's not just $M_\mu + M_\nu$!!

Compton Scattering

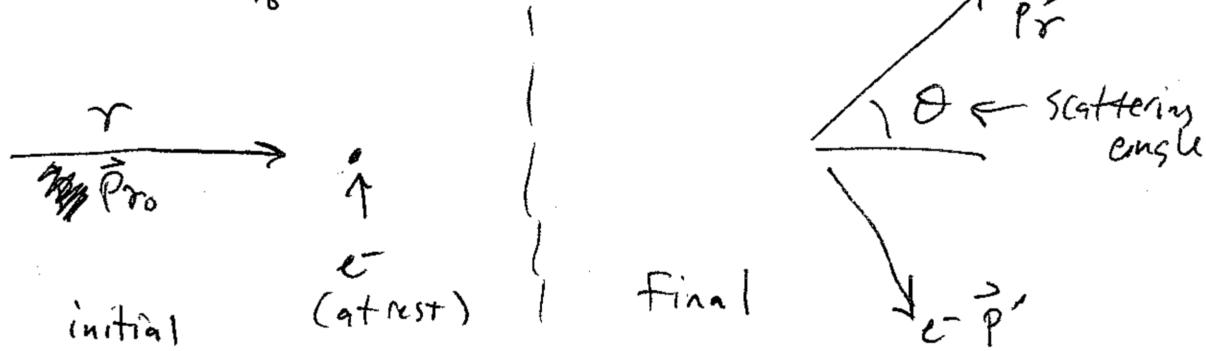
One of the most compelling pieces of evidence for the existence of photons - individual particles of light - comes from the scattering of photons on atomic electrons. The question is: do these photons really behave like small billiard balls, conserving momentum and energy in the collision?

The electron is considered to be at rest in the initial state, so its 4-momentum is

$$p_e = (\emptyset, \emptyset, \emptyset, m_e c) \quad \left(\text{or } p_e = (\emptyset, \emptyset, \emptyset, m_e) \right)$$

In practice the electron has a ~~small~~ non-zero expectation value to have a small amount of momentum while bound to the atom, but we neglect this in comparison with the energy and momentum of the incoming photon (gamma ray). The gamma ray 4-momentum is

$$p_{\gamma_0} = (\vec{p}_{\gamma_0}, E_{\gamma_0}/c)$$



According to the de Broglie hypothesis,

$$\vec{p}_0 = \hbar \vec{k}_0 \quad \text{and} \quad E = \hbar\omega_0, \text{ so}$$

$$p_{\gamma 0} = \left(\hbar \vec{k}_0, \frac{\hbar\omega_0}{c}\right) = \hbar\left(\vec{k}_0, \frac{\omega_0}{c}\right)$$

we can re-write in terms of unit vector $\hat{k}_0$:

$$|\vec{k}_0| = \frac{\omega_0}{c}, \quad \text{so} \quad \vec{k}_0 = \frac{\omega_0}{c}\hat{k}_0$$

Then

$$p_{\gamma 0} = \frac{\hbar\omega_0}{c}(\hat{k}_0, 1) \Leftarrow \text{initial gamma 4-momentum}$$

Then the final γ 4-momentum is

$$p'_\gamma = \frac{\hbar\omega'}{c}(\hat{k}', 1)$$

where $\omega' \neq \omega_0$, because the gamma has lost some energy.

The <u>final</u> electron 4-momentum is

$$p'_e = (\vec{p}', E'_e)$$

So we have

$$\underbrace{p_e + p_{\gamma 0} = p'_e + p'_\gamma}_{\text{conservation of 4-momentum}}$$

or $p'_e = p_e + (p_{\gamma 0} - p'_\gamma)$

Now we square both sides.

$$(p_e')^2 = p_e' \cdot p_e' = \left(p_e + (p_{\gamma_0} - p_\gamma')\right) \cdot \left(p_e + (p_{\gamma_0} - p_\gamma')\right)$$
$$= p_e^2 + 2p_e(p_{\gamma_0} - p_\gamma') + (p_{\gamma_0}^2 - 2p_{\gamma_0} \cdot p_\gamma' + p_\gamma'^2)$$

Now things simplify. We know that

$$\left.\begin{aligned} p_e' \cdot p_e' &= -m_e c \\ \text{and } p_e \cdot p_e &= -m_e c \end{aligned}\right\} \text{rest mass energy of the electron}$$

So these terms cancel.

$$\text{Also, } \left.\begin{aligned} p_{\gamma_0}^2 &= 0 \\ p_\gamma'^2 &= 0 \end{aligned}\right\} \text{rest mass energy of the photon.}$$

So we are left with

$$0 = 2p_e \cdot (p_{\gamma_0} - p_\gamma') - 2p_{\gamma_0} \cdot p_\gamma'$$

$$\text{or } p_{\gamma_0} \cdot p_\gamma = p_e \cdot (p_{\gamma_0} - p_\gamma') \qquad \boxed{\tfrac{\hbar}{c}(\omega_0 \hat{k}_0 - \omega' \hat{k}', \omega_0 - \omega')}$$

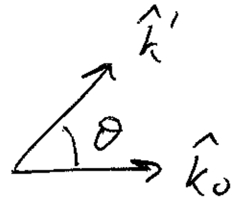
$$p_e = (0, 0, 0, m_e c)$$

$$= -m_e c \left(\tfrac{\hbar}{c}\right)(\omega_0 - \omega')$$

Left hand side:

$$p_{\gamma 0} \cdot p_{\gamma}' = \frac{\hbar \omega_0}{c}(\hat{k}_0, 1) \cdot \frac{\hbar \omega'}{c}(\hat{k}', 1)$$

$$= \left(\frac{\hbar}{c}\right)^2 \omega_0 \omega' \left[\hat{k}_0 \cdot \hat{k}' - 1 \right]$$



$$\hat{k}_0 \cdot \hat{k}' = \cos\theta$$

$$= \left(\frac{\hbar}{c}\right)^2 \omega_0 \omega' (\cos\theta - 1)$$

Therefore

$$\left(\frac{\hbar}{c}\right)^2 \omega_0 \omega' (\cos\theta - 1) = -m_e c \left(\frac{\hbar}{c}\right)(\omega_0 - \omega')$$

$$\boxed{\frac{\hbar}{m_e c^2}(1 - \cos\theta) = \frac{1}{\omega'} - \frac{1}{\omega_0}}$$

This relates the change in frequency to the scattering angle θ

In terms of wavelength, we have

$$\omega = \frac{2\pi c}{\lambda}$$

or

$$\boxed{\frac{h}{m_e c}(1 - \cos\theta) = \lambda' - \lambda_0}$$

This relates the change in wavelength to the scattering angle. Historically, the experimental confirmation of this formula confirmed the picture of a photon as a particle with $m = 0$.

Doppler Effect

Any plane wave can be written as

$$\phi = A\cos(\vec{k}\cdot\vec{x} - \omega t)$$

For a light wave, we also have

$$|\vec{k}| = \frac{2\pi}{\lambda}, \quad \text{and} \quad \omega = c|\vec{k}|$$

The phase of the plane wave must be the same in all frames of reference. For example, we can measure the phase by doing an interference experiment. All observers should agree whether the result is constructive interference or destructive interference, otherwise the laws of physics will be inconsistent from one frame to the next

Since $x = (\vec{x}, ct)$ is a 4-vector,

and since $\vec{k}\cdot\vec{x} - \omega t$ should have the same value in all frames of reference, we can define a new 4-vector

$$k \equiv (\vec{k}, \omega/c)$$

Then $k\cdot x = \vec{k}\cdot\vec{x} - \left(\frac{\omega}{c}\right)(ct) = \vec{k}\cdot\vec{x} - \omega t$

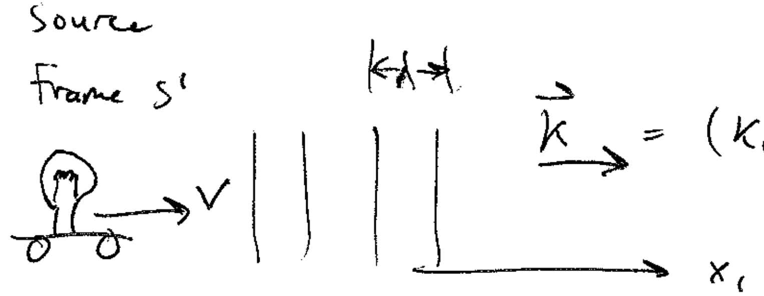
= Lorentz invariant phase.

We can derive the Doppler effect for light from the fact that k is a 4-vector.

According to the Lorentz Transformation,

$$k_4' = \frac{\omega'}{c} = |\vec{k}'| = \gamma(k_4 - \beta k_1)$$
$$= \gamma\left(\frac{\omega}{c} - \beta k_1\right)$$

For V along the direction of $\hat{k}$, we have



$\vec{k} = (k_1, 0, 0)$, $|\vec{k}| = k_1 = k$

Then

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \beta k\right) = \gamma\left(\frac{\omega}{c} - \beta\frac{\omega}{c}\right)$$
$$= \gamma\left(\frac{\omega}{c}\right)(1-\beta)$$

or $\omega' = \gamma\omega(1-\beta)$

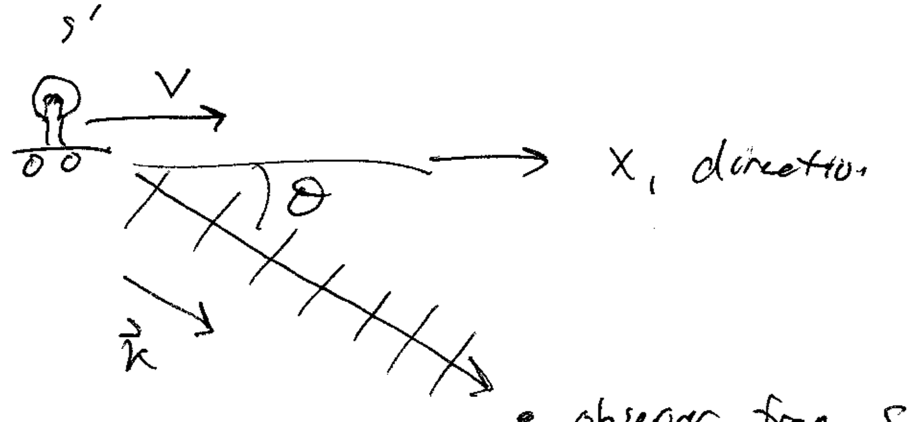
Let ω_0 be the frequency in frame S' (where the source is at rest). Then $\omega' = \omega_0$.

Solving for ω (frequency in frame S):

$$\boxed{\omega = \frac{\omega_0}{\gamma(1-\beta)}}$$

For ~~travel~~ relative motion in the direction of k.

In general the observer is located at some angle with respect to the motion:



Then $k_1 = |\vec{k}|\cos\theta$, and the result becomes

$$\boxed{\omega = \frac{\omega_0}{\gamma(1-\beta\cos\theta)}}$$

for observer at angle θ to the direction of motion.

Note that for $\theta = \emptyset$, we have

$$\omega = \frac{\omega_0}{\gamma(1-\beta)} = \frac{\omega_0}{\frac{(1-\beta)}{\sqrt{1-\beta^2}}} = \frac{\omega_0}{\frac{1-\beta}{\sqrt{(1-\beta)(1+\beta)}}} \boxed{= \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_0}$$

For $\theta = \pi/2$, the only effect is time dilation (no length contraction along the direction $\perp$ to the direction of motion.)

Then we have

$$\boxed{\omega = \frac{\omega_0}{\gamma}} \qquad (\theta = \pi/2)$$

This is simply time dilation.