Mechanics in Non-inertial Frames of reference.

This topic is important primarily because the earth's surface is a non-inertial frame of reference (due to the rotation of the earth.)

We can ~~do~~ analyze a mechanical system from the point of view of a non-inertial frame of reference, as long as we add the necessary pseudo-forces that make Newton's 2nd Law still hold. (effectively).

Let S_0 be an inertial frame and S be a frame that is accelerating with respect to S_0.

In S_0, we have $m\ddot{\vec{r}}_0 = \vec{F}$

In S, we have $\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$ ← velocity of S relative to S_0

velocity in S

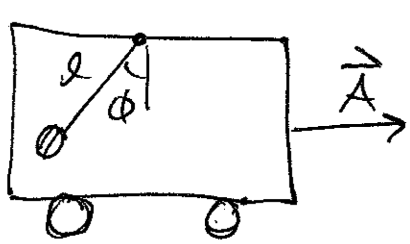
and $\ddot{\vec{r}} = \ddot{\vec{r}}_0 - \vec{A}$ ← acceleration of S relative to S_0

Therefore $m\ddot{\vec{r}} = \underbrace{m\ddot{\vec{r}}_0}_{\vec{F}} - m\vec{A}$

$$\boxed{m\ddot{\vec{r}} = \vec{F} - m\vec{A}}$$

So we can apply Newton's 2nd Law in the non-inertial frame, but we must add an additional force-like term ($-m\vec{A}$). This is called a "pseudoforce" or "fictitious force". ~~or~~ Pseudoforces are always proportional to the mass. Curiously, the gravitational force is also proportional to mass, which raises the question that gravity might be a pseudoforce. In fact, in general relativity, gravitational effects are essentially treated as an artifact of the choice of coordinate system, just like a pseudoforce.

Ex: Pendulum in an accelerating car

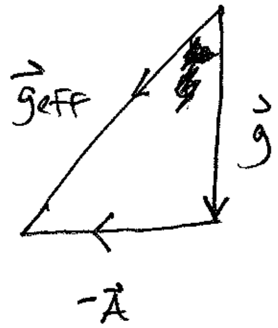


The forces are $\vec{T}$ (tension) and $m\vec{g}$.

From the perspective of a person inside the car, we also have a pseudoforce:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A} = \vec{T} + m(\vec{g} - \vec{A})$$
$$= \vec{T} + m\vec{g}_{eff}$$

where $\vec{g}_{eff} = \vec{g} - \vec{A}$



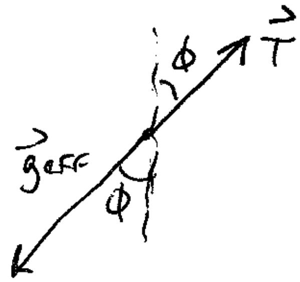
If the pendulum remains at rest (no oscillations), then

$$\vec{T} = -m\vec{g}_{eff}$$

And the direction of $\vec{T}$ tells us that

$$\phi_{equilibrium} = \tan^{-1}\left(\frac{A}{g}\right)$$

(because



The frequency of small oscillations is

$$\omega = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\sqrt{g^2+A^2}}{L}}$$

Tides

The earth and moon revolve around their common center-of-mass. Therefore ~~every drop of water on~~ the earth ~~surface accelerates towards the common CM, which point~~. accelerates towards the moon. We take the center of the earth as our origin and add a pseudoforce to account for the earth's acceleration.

Each drop of water experiences theses forces:

1) $m\vec{g}$ of the earth

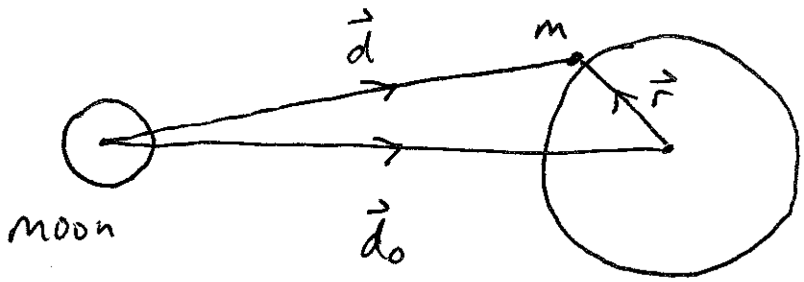
2) $-GM_m \frac{\hat{d}}{d^2}$, $\vec{d}$ points towards the moon, This is the gravitational force due to the moon.

3) $\vec{F}_{ng}$, a non-gravitational force such as the buoyant force. This holds the drop of water fixed in the earth's frame

4) A pseudoforce due to earth's acceleration

$$\vec{A} = -GM_m \frac{\hat{d}_o}{d_o^2}$$

, $\vec{d}_o$ is the position of earth's center relative to the moon's



The pseudo force is $-m\vec{A} = GM_m \frac{\hat{d}_o}{d_o^2}$

So we have

$$m\ddot{\vec{r}} = m\vec{g} - GM_m m \frac{\hat{d}}{d^2} + \vec{F}_{ng} + GM_m m \frac{\hat{d}_o}{d^2}$$

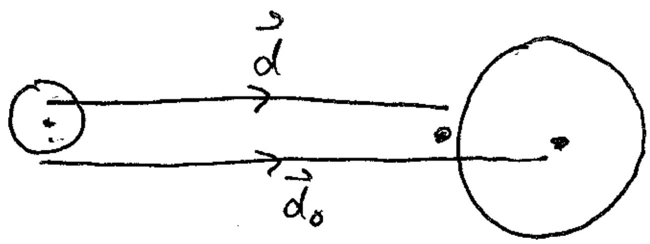
Define $\vec{F}_{tidal} \equiv -GM_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)$

Then $m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tid} + \vec{F}_{ng}$

In the absence of $\vec{F}_{tidal}$ this would be our normal equation of motion for any object on the earth surface (if the earth's surface were an inertial frame.)

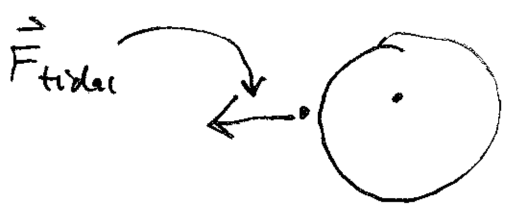
We can see the effect of the moon in $\vec{F}_{tidal}$

For a point near the moon:

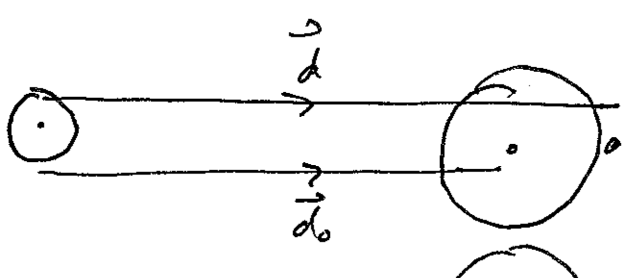


$\vec{d}$ is smaller than $\vec{d}_0$,

so



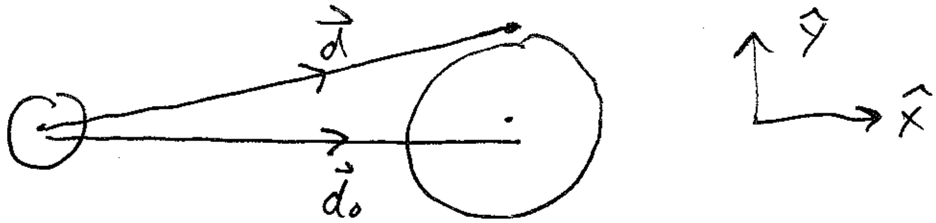
For a point opposite the moon:



$\vec{d}_0$ is smaller than $\vec{d}$,

so

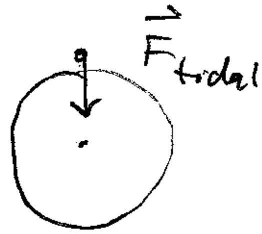
At 90^0 from the moon we have



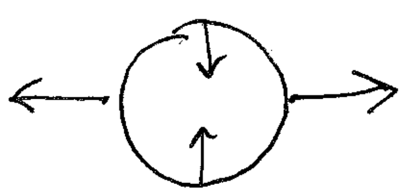
In $\vec{F}_{tidal\ y} = -GM_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)$

$\frac{\hat{d}_0}{d_0^2}$ has only an $\hat{x}$ component

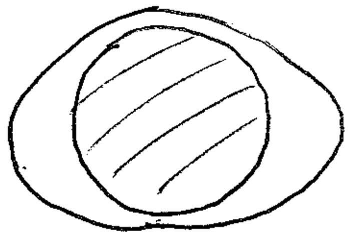
$\frac{\hat{d}}{d^2}$ has both x and y components. Because the moon is much further away than the radius of the earth, the x component of $\frac{\hat{d}}{d^2}$ will almost exactly cancel $\frac{\hat{d}_0}{d_0^2}$. This leaves only the $\hat{y}$ component:



So the total effect of $\vec{F}_{tidal}$ looks like



This gives the ocean 2 bulges.



and 2 ~~h~~ high tides per day as the earth rotates.

Rotating Frames of Reference.

For the case of a rotating frame, with angular velocity $\vec{\Omega}$, we have

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{V}$$

, $\vec{V}$ is velocity due to rotation of the frame

$= \vec{\Omega} \times \vec{r}$

so $$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}$$

We can consider this equation to be an operator which acts on $\vec{r}$:

$$\left(\frac{d}{dt}\right)_{S_0} = \text{operator} = \left[\left(\frac{d}{dt}\right)_S + \vec{\Omega} \times \right]$$

To get an expression for the acceleration as viewed from the rotating frame, we can apply the operator twice to $\vec{r}$:

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$$

$$\left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$
$$+ \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

Let's use the "dot" notation to describe time derivatives in the S system:

$$\dot{\vec{r}} = \left(\frac{d\vec{r}}{dt}\right)_S$$

Then

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \ddot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \frac{\vec{F}}{m} \quad \text{according to Newton's } 2^{nd} \text{ Law}$$

so

$$\boxed{m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}$$

where we have reversed the order of the cross products to get rid of minus signs.

The additional terms on the right hand side are pseudo forces. They have names:

$$\vec{F}_{coriolis} \equiv 2m\dot{\vec{r}}\times\vec{\Omega}$$

and $\vec{F}_{centrifugal} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$

For objects on the earth's surface they have magnitudes

$$|\vec{F}_{cor}| \sim mv\Omega$$

and $|\vec{F}_{CF}| \sim mr\Omega^2$, $r = R_{earth}$

so $$\frac{|\vec{F}_{cor}|}{|\vec{F}_{CF}|} \sim \frac{v}{R\Omega} \sim \frac{v}{V}$$

rotational velocity of earth's surface.

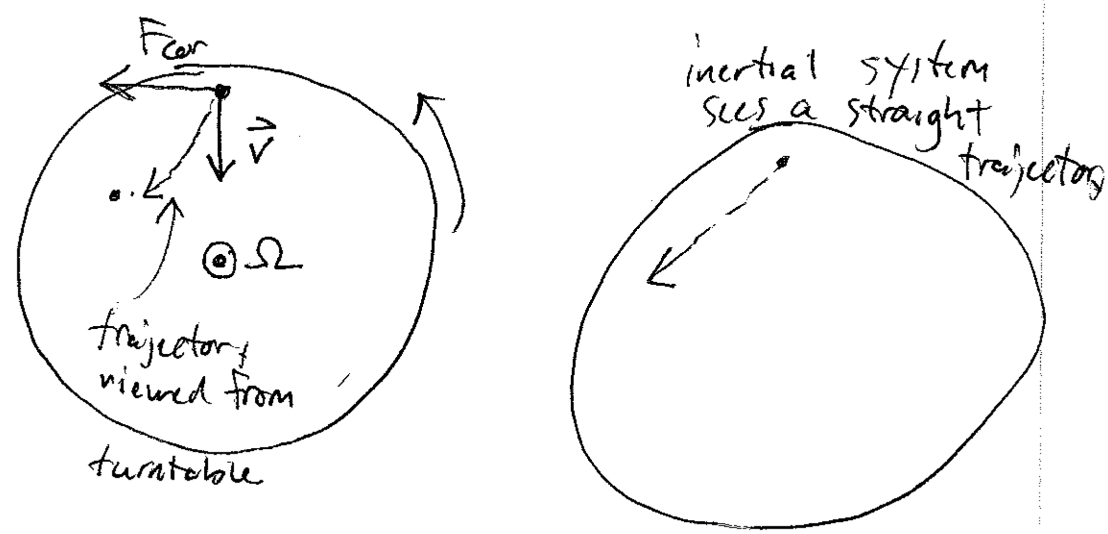
Earth rotational velocity at the surface (near the equator) is ~ 1000 miles/hour. so if the velocity of the object is small compared to this then we can probably ignore the coriolis force and can consider only the centrifugal force.

Coriolis Force

$$\vec{F}_{cor} = 2m\vec{v} \times \vec{\Omega}$$

This can be pictured as a magnetic-like force, with $2m \to q$ and $\vec{\Omega} \to \vec{B}$

On a turntable,

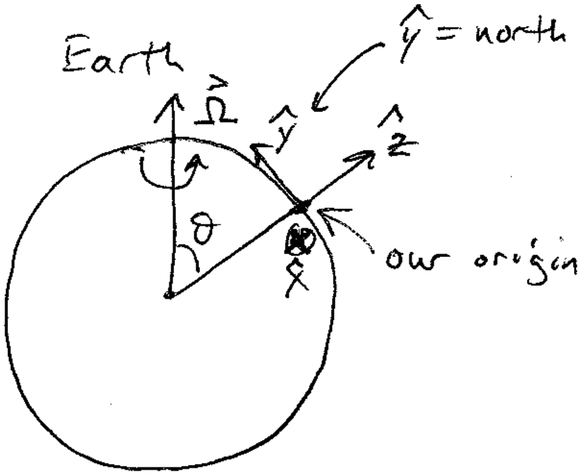


In the northern hemisphere, hurricanes rotate counterclockwise due to the coriolis effect.

Northern hemisphere

Southern hemisphere

Free fall with coriolis force



$$m\ddot{\vec r} = m\vec g + \underbrace{2m\dot{\vec r}\times\vec\Omega}_{\text{Coriolis force}}$$

small centrifugal force is included in $\vec g$.

$$\ddot{\vec r} = \vec g + 2\dot{\vec r}\times\vec\Omega \quad , \quad \vec g = -g\hat z$$

$$\dot{\vec r} = (\dot x, \dot y, \dot z), \quad \vec\Omega = (0, \Omega\sin\theta, \Omega\cos\theta)$$

so
$$\dot{\vec r}\times\vec\Omega = (\dot y\Omega\cos\theta - \dot z\Omega\sin\theta,\; -\dot x\Omega\cos\theta,\; \dot x\Omega\sin\theta)$$

so
$$\ddot x = 2\Omega(\dot y\cos\theta - \dot z\sin\theta)$$
$$\ddot y = -2\Omega\dot x\cos\theta$$
$$\ddot z = -g + 2\Omega\dot x\sin\theta$$

1st approximation: ignore Ω.

Then $\ddot{\vec r} = (0, 0, -g)$

$$\vec r = (0, 0, h - \tfrac{1}{2}gt^2)$$

2nd approximation

Take the previous solution and substitute:

$$\ddot{x} = 2\Omega g t \sin\theta$$
$$\ddot{y} = 0$$
$$\ddot{z} = -g$$

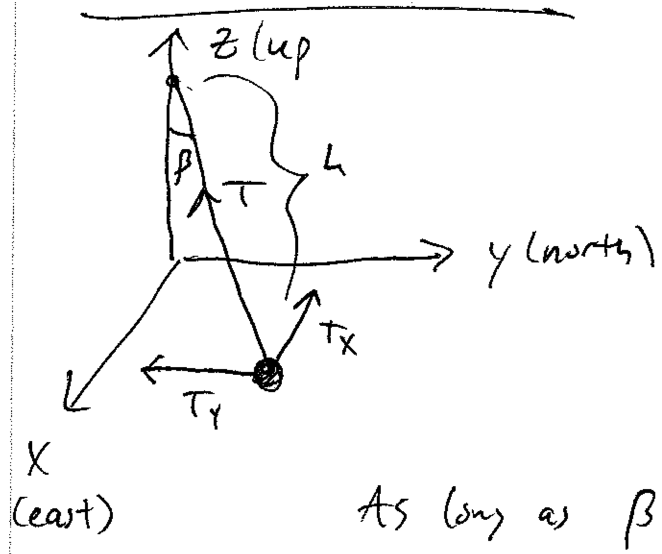
Then $\boxed{x = \frac{1}{3}\Omega g t^3 \sin\theta}$

So the object is deflected in the $(+x)$ direction. If the object falls 100 m, without drag, at the equator then $t = \sqrt{2h/g}$ ≈ and

$$x = \frac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2}, \quad \Omega = 7.3\times10^{-5}\ \text{sec}^{-1}$$

$$x = \frac{1}{3}(7.3\times10^{-5})(10)(20)^{3/2} \approx 2.2\ \text{cm}$$

Foucalt Pendulum



$$m\ddot{\vec{r}} = \vec{T} + 2m\dot{\vec{r}}\times\vec{\Omega} + m\vec{g}$$

includes centrifugal term

As long as β is small,

$$T_z \approx |\vec{T}| \approx mg \approx T$$

By similar triangles,

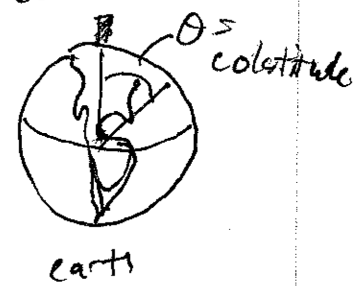
$$\frac{T_x}{T} = -\frac{x}{L} \quad \text{and} \quad \frac{T_y}{T} = -\frac{y}{L}$$

$$\Rightarrow T_x = -\frac{mgx}{L}, \quad T_y = -\frac{mgy}{L}$$

$$\ddot{x} = -\frac{gx}{L} + 2\dot{y}\Omega\cos\theta$$

$$\ddot{y} = -\frac{gy}{L} - 2\dot{x}\Omega\cos\theta$$

, θ = colatitude



$$\frac{g}{L} = \omega_0^2, \quad \Omega\cos\theta = \Omega_z$$

$$\boxed{\begin{aligned} \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x &= \emptyset \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y &= \emptyset \end{aligned}}$$

<u>Coupled differential equations</u>: Define $\eta = x + iy$

Multiply 2nd Equation by (i) and add:

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = \emptyset$$

Guess solution of the form $\eta(t) = e^{-i\alpha t}$

Then $\alpha^2 - 2\Omega_z\alpha - \omega_0^2 = \emptyset$

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} \approx \Omega_z \pm \omega_0$$

General Solution:

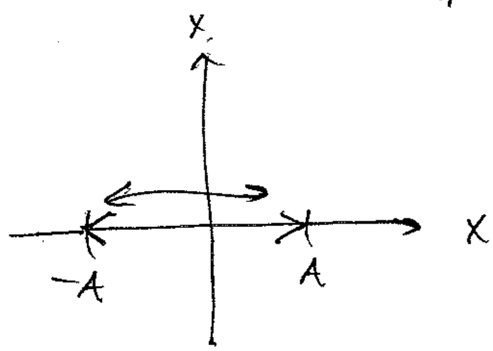
$$\eta = e^{-i\Omega_z t}\left(c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right)$$

Make up some initial conditions:

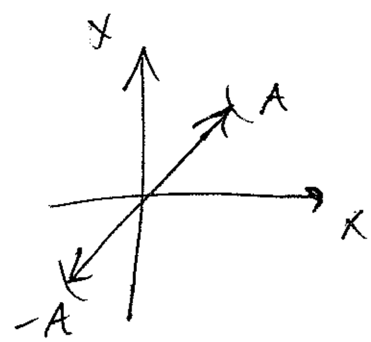
$$x(t=0) = A$$
$$y(t=0) = 0$$
$$v_x(t=0) = 0$$
$$v_y(t=0) = 0$$

Then $\eta(t) = x(t) + i y(t) = A e^{-i\Omega_z t}\cos(\omega_0 t)$

Since Ω_z is small, initially the oscillation is entirely in the x direction:



but after a while



The rate of rotation is $\Omega\cos\theta$, $\theta = 51.0°$ for College Park, so $\cos\theta \approx 63\%$, so the full period in College Park $\approx$ 1.59 days.