Phys 410 – Homework #5

Numbered problems are from Taylor.

1) Taylor 6.1 (3 pts)
2) Taylor 6.16 (3 pts)
3) *A minimum in the action, or a saddle point? (9 pts total).*

a) Write down the Lagrangian for a simple harmonic oscillator (mass on a spring) without drag forces in terms of x, x-dot, m, and k. Let $x_0(t)$ denote a true path of the oscillator. We now consider variations on this path of the form $x_0(t) + \xi(t)$, where $\xi(t)$ goes to zero at t = 0 and $t = t_1$. If $S[\xi]$ represents the action for the variation ξ, show that

$$S[\xi] = \int_0^{t1} \left(\frac{m}{2}\left(\dot{x}_0^2 + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + \xi^2\right) \right) dt .$$

Hint: you will have cross terms involving x_0, ξ, and their first time derivatives. Use integration by parts and the fact that x_0 satisfies the equation of motion to eliminate these terms.

b) Consider whether the variation $\xi(t)$ increases or decreases the action in the neighborhood of $x_0(t)$. Let $S_0 = S[\xi=0]$, the action for the true path, and let $\Delta S = S[\xi] - S_0$, so that ΔS is the change in the action due to variation $\xi(t)$. Then we have

$$\Delta S = S[\xi] - S_0 = \frac{1}{2}\int_0^{t1} \left(m\dot{\xi}^2 - k\xi^2 \right) dt$$

Let's choose a simple triangle function for the variation:

$$\xi(t) = \begin{cases} \dfrac{\varepsilon t}{t_1}, & 0 \le t \le \dfrac{t_1}{2} \\[2ex] \varepsilon\left(1 - \dfrac{t}{t_1}\right), & \dfrac{t_1}{2} \le t \le t_1 \end{cases}$$

Find the condition for t_1 under which ΔS is negative (where the variation has decreased the action), and compare this value of t_1 to the full period of the oscillator. *Remark*: Since the triangle function variation can *decrease* the action for certain values of t1, this shows that the true path $x_0(t)$ can be *a saddle point* in the (not a minimum).

c) Repeat part (b) for a variation of the type $\xi(t) = \varepsilon \sin(\pi t / t_1)$