Phys 410 – Homework #5

Numbered problems are from Taylor. Note that this assignment has two pages.

1) Taylor 6.1 (3 pts)
2) Taylor 6.16 (3 pts)
3) *A minimum in the action, or a saddle point? The case of the simple harmonic oscillator (9 pts total).*

a) Write down the Lagrangian for a simple harmonic oscillator (mass on a spring) in terms of x, x-dot, m, and k. Let $x_0(t)$ denote a true path of the oscillator, so that $x_0(t)$ satisfies the SHO equation of motion. We now consider variations on this path of the form $x_0(t) + \xi(t)$, where $\xi(t)$ goes to zero at t = 0 and $t = t_1$. If $S[\xi]$ represents the action for the variation ξ, show that

$$S[\xi] = \int_0^{t1} \left(\frac{m}{2}\left(\dot{x}_0^2 + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + \xi^2\right) \right) dt \,.$$

Hint: you will have cross terms involving x_0, ξ, and their first time derivatives. Use integration by parts and the fact that x_0 satisfies the equation of motion to eliminate these terms.

b) Let's assume that the true path $x_0(t)$ represents a stationary point in the action. (In fact, it is a stationary point, as required by Hamilton's Principle.) We would like to understand is whether $x_0(t)$ is a minimum in the action or a saddle point in the action. To address this question, consider whether the variation $\xi(t)$ increases or decreases the action in the neighborhood of $x_0(t)$. As always, we will only consider fixed time intervals, in this case the time interval from t = 0 to $t = t_1$. Let $S_0 = S[\xi=0]$, the action for the true path, and let $\Delta S = S[\xi] - S_0$, so that ΔS is the change in the action due to variation $\xi(t)$. Then we have

$$\Delta S = S[\xi] - S_0 = \frac{1}{2}\int_0^{t1} \left(m\dot{\xi}^2 - k\xi^2 \right) dt$$

Let's choose a simple triangle function for the variation:

$$\xi(t) = \begin{cases} \dfrac{\varepsilon t}{t_1}, & 0 \le t \le \dfrac{t_1}{2} \\[2ex] \varepsilon\left(1 - \dfrac{t}{t_1}\right), & \dfrac{t_1}{2} \le t \le t_1 \end{cases}$$

Find the condition for t_1 under which ΔS is negative (where the variation has decreased the action), and compare this value of t_1 to the full period of the oscillator.

(continued on next page).

Remark: We can always *increase* the action around the true path by increasing the KE term with a high-frequency, small-amplitude wiggle. Since the example of the triangle function shows that it is also possible to find variations that *decrease* the action (at least in some situations), this shows that the true path $x_0(t)$ represents *a saddle point* in the action for those cases, not a minimum. In other words, the action may increase or decrease around the true path, depending on the exact nature of the variation that is considered, and depending on the choice of t_1. This is why Hamilton's Principle is 'the principle of stationary action' and not 'the principle of least action'.

c) Repeat part (b) for a variation of the type $\xi(t) = \varepsilon \sin(\pi t / t_1)$