an integral of the form

$$\int_{x_1}^{x_2} F\left[y(x), y'(x), x\right] dx$$

is solved by the Euler Lagrange equation

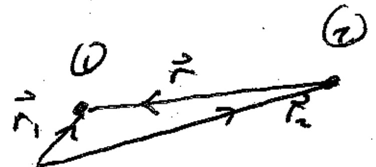
$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad \text{where } y' = \frac{dy}{dx}$$

For example, the length of a curve $y(x)$ in the xy plane is given by

$$\int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx$$

To find the minimum length between x_1 & x_2 we solved $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ for $F = \sqrt{1+(y')^2}$.

<u>Central Force Motion</u>



$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \text{let } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

$$T = \tfrac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right), \quad \mu = \frac{m_1 m_2}{m_1+m_2} = \text{“reduced mass”}$$

$$\mathcal{L} = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{cm}} + \underbrace{\left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{\mathcal{L}_{relative}}$$

Use Polar Coordinates for the relative motion, with the origin at the CM:

$$\mathcal{L}_{rel} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

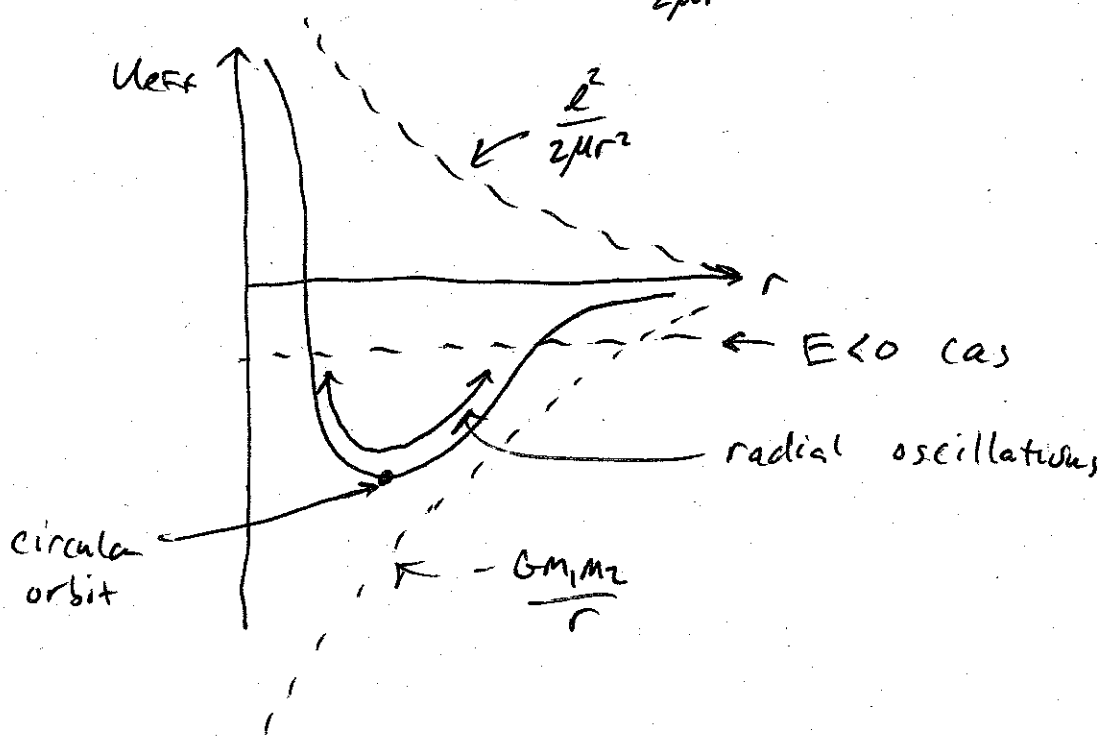
$$\Rightarrow \dot{\phi} = \frac{\ell}{\mu r^2}, \quad \ell = \text{constant} \quad (\text{angular momentum})$$

Radial Equation:

$$\mu\ddot{r} = -\frac{d}{dr}\underbrace{\left[U(r) + \frac{\ell^2}{2\mu r^2}\right]}_{U_{eff}(r)}$$

Energy conservation: $\tfrac{1}{2}\mu\dot{r}^2 + U_{eff} = \text{constant} = E.$

For gravity, $U_{eff} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$



Orbital Equation: (determining the shape of the orbit)

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u^2(\phi)} F \quad \text{where } u \equiv \frac{1}{r}$$

For the case of the inverse square law,

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2 \quad , \gamma = G m_1 m_2$$

Then the orbital equation is solved by

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$

where $c = \dfrac{\ell^2}{\gamma\mu}$ = units of length

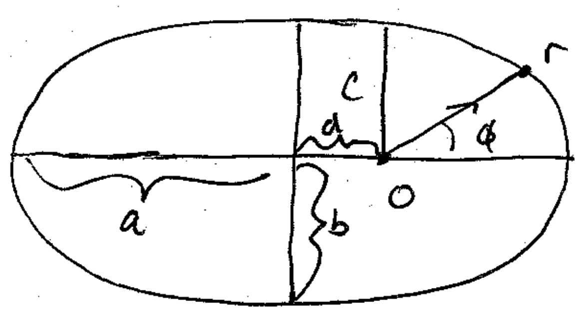
ε = eccentricity : $\varepsilon = 0$ (circular orbit)

$0 < \varepsilon < 1$ (elliptical)

$\varepsilon = 1$ (parabolic)

$\varepsilon > 1$ (hyperbolic)

For an elliptical orbit,



$$a = \frac{c}{1-\varepsilon^2}$$

$$b = \frac{c}{\sqrt{1-\varepsilon^2}}$$

$$d = a\varepsilon$$

Also, $\varepsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$

Kepler's Third Law: $\tau^2 = \frac{4\pi^2 \mu a^3}{\gamma}$ Semi-major axis length

τ = period

Relationship between energy and ε and ℓ:

$$E = \frac{\gamma^2 \mu}{2\ell^2}\left(\varepsilon^2 - 1\right)$$

Angular change: $\phi(r) = \int_{r_1}^{r_2} \frac{\frac{\ell}{r^2}\, dr}{2\mu\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)}$

Angular Momentum and Rigid Bodies

For a fixed rotation axis, and ignoring the components of $\vec{L}$ that are $\perp$ to the rotation axis, we have

$$L_z = I_z \omega \quad , \quad I_z = \int \left(x^2 + y^2\right) dm = \int r^2\, dm$$
$$= \sum_i m_i r_i^2$$

Then $T = \frac{1}{2} I_z \omega^2$

Including the motion of the CM,

$$\vec{L} = \vec{R}_{cm} \times \vec{P} + \left(I_z^{cm} \omega'\right) \hat{z}$$

rotational angular velocity about the center of mass.

If the rotation axis is not fixed, and/or we desire to know about all 3 components of $\vec{L}$, then we have

$$\vec{L} = I\vec{\omega}, \quad \vec{\omega} = \text{angular velocity vector}$$
$$I = \text{inertia tensor}$$

$$I = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)dm \end{pmatrix}$$

Kinetic Energy: $T = \frac{1}{2}\vec{\omega}\, I\, \vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

Principal Axes: We can always find at least 3 orthogonal directions in space (unit vectors) for which

$$I\hat{e}_1 = \lambda_1 \hat{e}_1 \qquad \hat{e}_1, \hat{e}_2, \hat{e}_3 \text{ are eigenvectors}$$
$$I\hat{e}_2 = \lambda_2 \hat{e}_2 \qquad \lambda_1, \lambda_2, \lambda_3 \text{ are the eigenvalues}$$
$$I\hat{e}_3 = \lambda_3 \hat{e}_3$$

We call the eigenvectors the principal axes, and the $\{\lambda\}$, the principal moments.

If we choose the principal axes as the coordinate system, then I will appear as a diagonal matrix:

$$I = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

In the system of the principal axes,

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$\omega_1, \omega_2, \omega_3$ are the projections of $\vec{\omega}$ onto the principal axes at each moment.

If we project $\frac{d\vec{L}}{dt}$ and $\vec{\omega}$ onto the principal axes at each moment (even though the principle axes keep rotating), we have

$$\left(\frac{d\vec{L}}{dt}\right)_1 = \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2$$

$$\left(\frac{d\vec{L}}{dt}\right)_2 = \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3$$

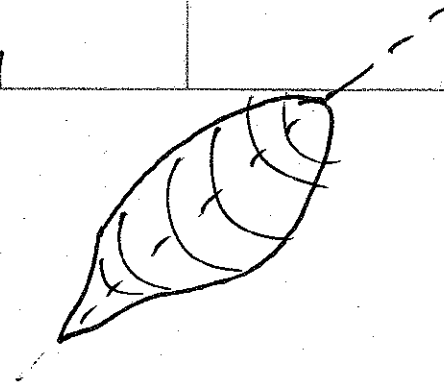
$$\left(\frac{d\vec{L}}{dt}\right)_3 = \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1$$

And since $\vec{\Gamma} = \frac{d\vec{L}}{dt}$ we have

$$\left.\begin{aligned}\Gamma_1 &= \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2\\ \Gamma_2 &= \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3\\ \Gamma_3 &= \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1\end{aligned}\right\} \text{ Euler's Equations.}$$

These are most useful for the Free precession of a top, when $\Gamma_1 = \Gamma_2 = \Gamma_3 = \emptyset$.

Symmetric Free Top:

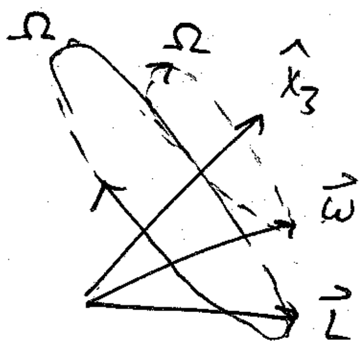


$$I = \begin{pmatrix} I & \emptyset & \emptyset \\ \emptyset & I & \emptyset \\ \emptyset & \emptyset & I_3 \end{pmatrix}$$

Then $\vec{\omega} = \underbrace{(\omega_0 \cos(\Omega t), -\omega_0 \sin(\Omega t), \omega_3)}$

projected onto the rotating body axes

where $\Omega = \left(\dfrac{I - I_3}{I}\right) \omega_3$



$\hat{x}_3$, $\vec{\omega}$, and $\vec{L}$ remain coplanar. $\vec{\omega}$ & $\vec{L}$ precess about $\hat{x}_3$ as viewed from the body frame

Final Exam Review

Accelerating Frames of Reference:

$$m\ddot{\vec{r}} = \vec{F} \underbrace{- m\vec{A}}_{\uparrow \text{"pseudo-force"}} , \quad \vec{A} \text{ is acceleration of the frame of reference w/respect to an inertial system}$$

Rotating Frames, described by $\vec{\Omega}$ vector:

Two pseudoforces:

$$\vec{F}_{coriolis} \equiv 2m\dot{\vec{r}} \times \vec{\Omega}$$

$$\vec{F}_{centrifugal} \equiv m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

Hamiltonian Mechanics

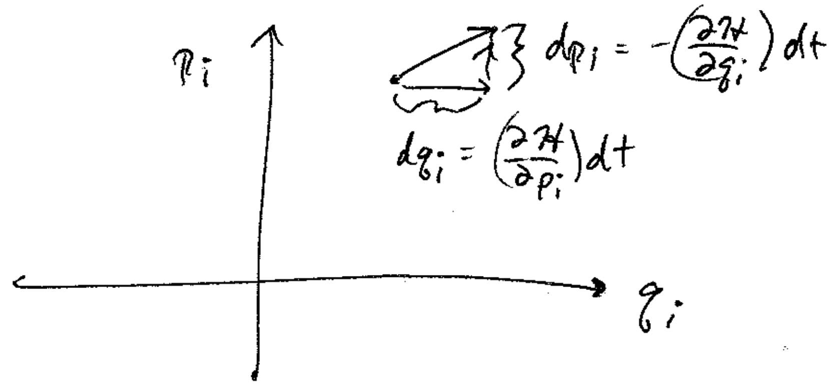
$p_i \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$, Then $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$, $i = 1, 2, 3, \ldots$ for each generalized coordinate.

Equations of Motion:

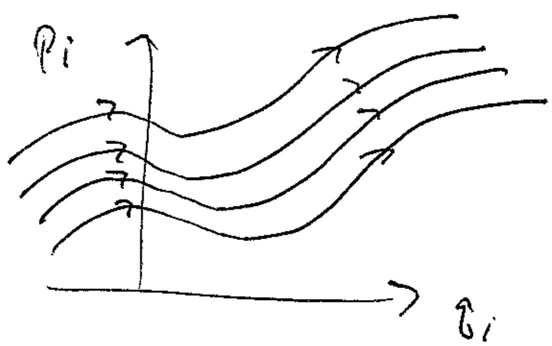
$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned}}$$

Conservation of Energy: The Hamiltonian is constant if the Lagrangian has no explicit time dependence. Also, the value of the Hamiltonian is exactly equal to the energy if the potential is velocity independent and the equations connecting the q_i and (x, y, z) do not depend on time.

Phase Space:



There can be no crossings in phase space.



Phase space density is conserved according to Liouville's Theorem

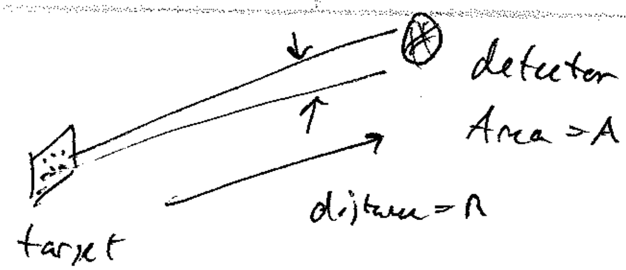
Rutherford Scattering

$$N_{scattered} = (N_{incoming})(n_{target})\,\sigma$$

n_{target}: # of target per unit volume

σ: cross-section, units = m^2 or cm^2 or barn ($10^{-24} cm^2$)

Solid Angle:



$$\Delta\Omega = \frac{A}{R^2}$$

Limit when $A \to \emptyset$, then $\Delta\Omega \to d\Omega = \frac{dA}{r^2}$

$$= \sin\theta \, d\theta \, d\phi$$

$$d\sigma(\text{scattering into } d\Omega) = \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

$\frac{d\sigma}{d\Omega}$ = differential cross section, $\sigma_{total} = \int \frac{d\sigma}{d\Omega} \, d\Omega$

$$= \int_\emptyset^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \frac{d\sigma}{d\Omega}(\theta,\phi)$$

If b is the impact parameter which causes scattering angle θ, then

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For Rutherford scattering (inverse square law),

$$b(\theta) = \frac{\gamma}{mv^2}\cot\left(\frac{\theta}{2}\right), \quad \gamma = kqQ$$

Then $$\frac{d\sigma}{d\Omega} = \left(\frac{kqQ}{4E\sin^2(\theta/2)}\right)^2$$