Phys 410 | Week 6 – Exam Review (1)

Differential Equations

- $\frac{dy}{dt} = A \Rightarrow y(t) = At + C$
- $\frac{dy}{dt} = Ay \Rightarrow y(t) = Ce^{At}$
- $\frac{d^2y}{dt^2} + Ay = 0 \Rightarrow y(t) = c_1 \sin(\sqrt{A}\,t + c_2)$ (and other ways to write it.)

Euler Method: (Numerical)

$$v_{i+1} = v_i + a_{i+1}\Delta t$$

$$x_{i+1} = x_i + v_{i+1}\Delta t$$

$$a_{i+1} = \text{equation of motion}$$

1 Dimensional Single Particle Mechanics:

IF $F = F(t)$, then $v(t) = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t')\,dt'$

$$x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'$$

IF $F = F(x)$, then use $\cancel{a = \frac{dv}{dt}}$ $a = v\frac{dv}{dx}$, or

$$v(x) = \left[v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'\right]^{\frac{1}{2}}$$

and $t = t_0 + \int_{x_0}^{x(t)} \frac{dx'}{v(x')}$

IF $F = F(v)$, then

$t = t_0 + m\int_{v_0}^{v(t)} \frac{dv'}{F(v')}$, then solve for $v(t)$. and integrate to get $x(t)$.

Newtons Laws:

2) $\vec{F} = m\vec{a} = m\ddot{\vec{r}} = \dot{\vec{p}}$

3) $\vec{F}_{12} = -\vec{F}_{21}$ $\Rightarrow$ leads to conservation of momentum

Polar coordinates

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r}$$, and so

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

Air resistance

$$\vec{F}_v = -b\vec{v} \quad \text{(linear drag)}$$

or $\vec{F}_v = -cv^2\hat{v}$ or $\vec{F}_v = -cv\vec{v}$

$$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x$$

$$m\dot{v}_y = mg - c\sqrt{v_x^2 + v_y^2}\, v_y$$

with gravity

Charged particle in uniform field: $\omega = qB/m$

$$\dot{v}_x = \omega v_y$$
$$\dot{v}_y = -\omega v_x$$

$\Rightarrow$ $\xi = x + iy = \sim e^{-i\omega t}$

while $z(t) = z_0 + v_{0z}t$

Conservation of Momentum

$$\dot{\vec{P}}_{total} = \vec{F}_{external} \Rightarrow \vec{P}_{total} = \text{constant}$$

when $\vec{F}_{external} = \emptyset$.

Rocket Motion: $\boxed{m(t)\dot{v} = -\dot{m}(t)V_{ex}}$

Center of Mass: $M \equiv \sum_\alpha m_\alpha$ = total mass

$$\vec{R}_{CM} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{1}{M}\int_{\text{volume}} \rho\,\vec{r}\,dV$$

Angular Momentum: (single particle)

$$\vec{\ell} = \vec{r} \times \vec{p}$$

$$\dot{\vec{\ell}} = \vec{r} \times \vec{F} \equiv \vec{\Gamma}$$

$\boxed{\dot{\vec{\ell}} = \vec{\Gamma}}$ Newton's 2nd Law in angular form.

Total Angular Momentum of a system of particles:

$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha$. Then $\dot{\vec{L}} = \vec{\Gamma}^{ext}$.

Moment of Inertia

$$I \equiv \sum_\alpha m_\alpha \rho_\alpha^2 = \int \rho(x,y,z)\,\rho_\alpha^2\,dV$$

ρ_α: distance to rotation axis; $\rho(x,y,z)$: density function; ρ_α: distance to axis.

For a fixed rotation axis, $L_z = I\omega$

or $I\dot{\omega} = \Gamma_z$

Work & Energy

$$dT = \vec{F}\cdot d\vec{r} \Rightarrow \Delta T = \int_1^2 \vec{F}\cdot d\vec{r}$$

If $\vec{\nabla}\times\vec{F} = \emptyset$, then the work of $\vec{F}$ is path independent, and we say it is conservative. We can find a function U such that

$$U(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

Then $\Delta T + \Delta U = \Delta(T+U) \equiv \Delta E = \emptyset$
(energy is conserved.)

In One-Dimension

$$W_{12} = \int_{x_1}^{x_2} F(x')\,dx' \quad , \quad U(x) = -\int_{x_0}^{x} F(x')\,dx' \,,$$

$F_x = -\frac{dU}{dx}$. For a conservative system,

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E-U(x)} \quad \text{or}$$

$$\boxed{t_f = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-U(x')}}}$$