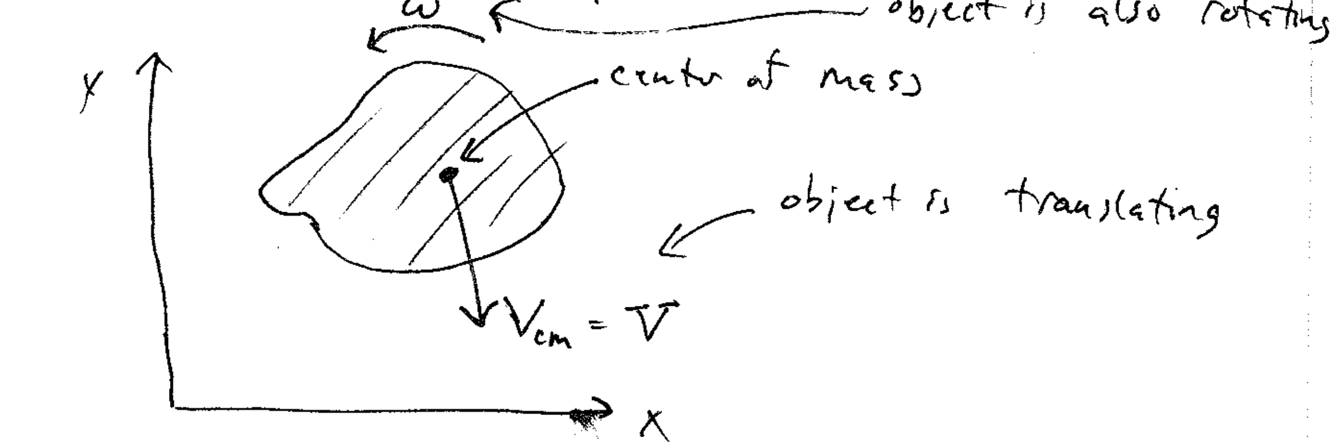
Angular momentum and ~~Rigid~~ Rigid body rotations.

We'll start with rotations around a fixed axis, which are rather simple (and usually covered in introductory physics.) As we'll see, when the rotation axis is no longer fixed, the behavior of the rotating object can be quite complicated. This is one of the most conceptually difficult topics in classical mechanics.

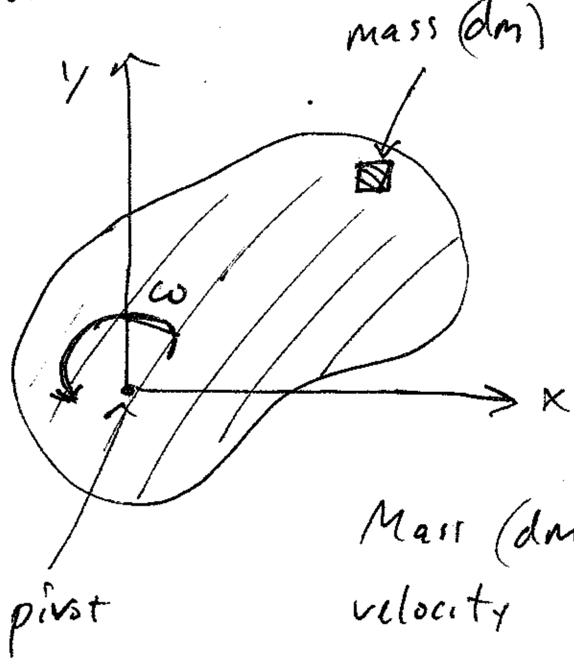
Pancake object in the xy plane

First, recall that angular momentum can only be defined relative to a specific origin. It makes no sense to ask what $\vec{L}$ is without specifying the origin.

For a flat object lying in the xy plane and moving in that plane:



Special case: The pancake pivots about the ~~origin~~ z-axis in the counter-clockwise direction.



$$L = \sum_i \vec{r}_i \times \vec{p}_i$$

↖ sum over all particles

Mass (dm) at position (x, y) has velocity $v = \omega r = \omega (x^2+y^2)^{\frac{1}{2}}$.

Its angular momentum is

$$d\vec{L} = \vec{r} \times d\vec{p} = r v\, dm\, \hat{z}$$
$$= r\,(r\omega)\, dm\, \hat{z}$$
$$= dm\, r^2 \omega\, \hat{z}$$

The total angular momentum is

$$\vec{L} = \int r^2 \omega\, \hat{z}\, dm = \int (x^2+y^2)\,\omega\, \hat{z}\, dm$$

If density is constant (ρ), then $dm = \rho\, dx\, dy$

Then

$$\vec{L} = \int (x^2+y^2)\,\omega \rho\, \hat{z}\, dx\, dy$$

We can write it more compactly by defining

The "moment of inertia about the z axis"

$$\boxed{I_z \equiv \int (x^2+y^2)\, dm} = \boxed{\int r^2 dm}$$

Then the z-component of $\vec{L}$ is

$$\boxed{L_z = I_z \omega} \leftarrow \text{analogous to } \quad p_z = m v_z$$

and $L_x = L_y = \emptyset$.

For a discrete set of particles,

$$I_z \equiv \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$

For any rigid body in the xy plane we can calculate I_z. Then for a rotation around the fixed z axis, we multiply I_z by ω to get L_z.

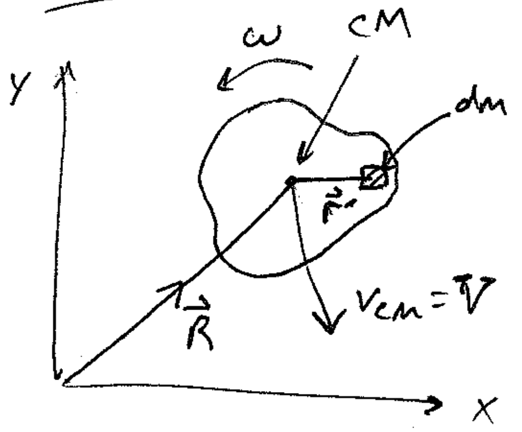
Kinetic Energy: $dT = dm \frac{v^2}{2} = dm \frac{(r\omega)^2}{2}$

$$T = \int dT = \int \frac{r^2\omega^2}{2} dm$$

Using our definition of I_z, we have

$$\boxed{T = \tfrac{1}{2} I_z \omega^2} \leftarrow \text{analogous to } T = \tfrac{1}{2} m v^2.$$

General Motion in the xy plane, including translation:



position of dm:
$$\vec{r} = \vec{R} + \vec{r}'$$
Its velocity:
$$\vec{v} = \vec{V}_{cm} + \vec{v}'$$
↑ velocity relative to center-of-mass

Let the body rotate with angular velocity ω' around the center-of-mass (axis in the z direction). Then $\vec{v}' = \omega' r'$. Then

$$\vec{L} = \int \vec{r} \times \vec{v}\, dm = \int (\vec{R} + \vec{r}') \times (\vec{V}_{cm} + \vec{v}')\, dm$$
$$= \int (\vec{R} \times \vec{V}_{cm})\, dm$$
$$+ \int (\vec{r}' \times \vec{v}')\, dm$$

These two are zero: ⟶
$$\left\{ \begin{array}{l} + \int (\vec{r}' \times \vec{V}_{cm})\, dm \\ + \int (\vec{R} \times \vec{v}')\, dm \end{array} \right.$$

(For example, $\int \vec{r}' \times \vec{V}_{cm}\, dm = -\vec{V}_{cm} \times \underbrace{\int \vec{r}'\, dm}_{\text{coordinate of center of mass relative to the center of mass}} = \emptyset$)

(Similarly, $\vec{R} \times \int \vec{v}'\,dm = \vec{R} \times \frac{d}{dt}\int \vec{r}'\,dm = \phi$)

so $$\vec{L} = \int (\vec{R} \times \vec{V}_{cm})\,dm + \int (\vec{r}' \times \vec{v}')\,dm$$
$$= (\vec{R} \times \vec{V}_{cm}) \int dm + \left(\int r'^2 \omega'\,dm\right)\hat{z}$$
$$= (\vec{R} \times M\vec{V}_{cm}) + (I_z^{cm}\,\omega')\hat{z}$$

where I_z^{cm} is the moment of inertia around an axis parallel to z but passing through the center of mass. Summarizing,

$$\boxed{\vec{L} = \underbrace{\vec{R} \times \vec{P}}_{\vec{L}\text{ of the center-of-mass}} + \underbrace{(I_z^{cm}\,\omega')\hat{z}}_{\vec{L}\text{ about the center-of-mass}}}$$

Kinetic Energy:

$$T = \int \tfrac{1}{2} v^2\,dm = \int \tfrac{1}{2}(\vec{V}_{cm} + \vec{v}')^2\,dm$$
$$= \tfrac{1}{2}\int V_{cm}^2\,dm + \tfrac{1}{2}\int v'^2\,dm$$

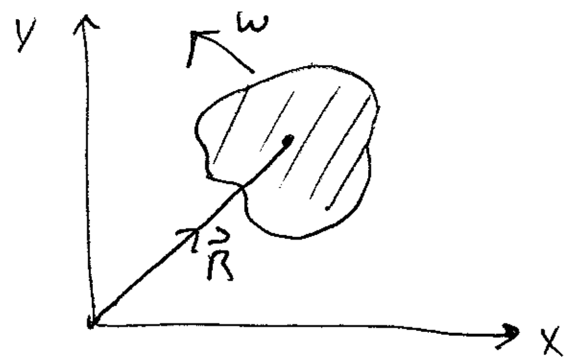
+ cross terms which vanish for the same reason as above

$$= \tfrac{1}{2} M V_{cm}^2 + \tfrac{1}{2}\int r'^2 \omega'^2\,dm$$

$$\boxed{T = \tfrac{1}{2} M V_{cm}^2 + \tfrac{1}{2} I_z^{cm}\,\omega'^2}$$

Parallel Axis Theorem

Move the axis of rotation from the CM to the origin:



All points are travelling in a circle about the origin. Let the angular speed be ω. Then the speed of the CM is $V_{cm} = \omega R$, and according to our previous calculation of $\vec{L}$, we have

$$L_z = RP + I_z^{cm}\, \omega_{cm}$$

angular velocity about CM

$$= RMV_{cm} + I_z^{cm}\, \omega_{cm}$$

$$= RM(\omega R) + I_z^{cm}\, \omega_{cm}$$

$$= MR^2\omega + I_z^{cm}\, \omega_{cm}$$

But ω_{cm} must equal ω, so that the pancake returns to its original orientation after one full revolution. So

$$L_z = \left(MR^2 + I_z^{cm}\right)\omega$$

or $L_z = I_z\omega$, where $\boxed{I_z = MR^2 + I_z^{cm}}$

Non-planar objects rotating about a fixed z-axis:

In the case that the object has a finite size in the z-direction (not a simple pancake), our results for the pancake are still valid provided that

1) the axis of rotation is fixed,
2) it's parallel to z, and
3) we do not care about L_x and L_y (they may no longer be zero.)

For example, we can imagine slicing an object into z slices, and for each z slice we will have $I_z = \int (x^2+y^2)\, dm$ and $L_z = I_z \omega$.

Since I_z does not depend on z, the same expressions will work for the sum of the z slices.

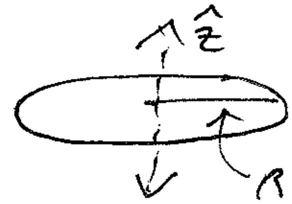
Note that L_x and L_y are not necessarily zero with the non-planar object.

Moment of Inertia calculations.

Ex: A ring of mass: total mass M, radius R:

$$I = \int r^2 dm = \int_0^{2\pi} R^2 (\rho R\, d\theta)$$

$$= \underbrace{(2\pi R\rho)}_{M} R^2 = MR^2$$

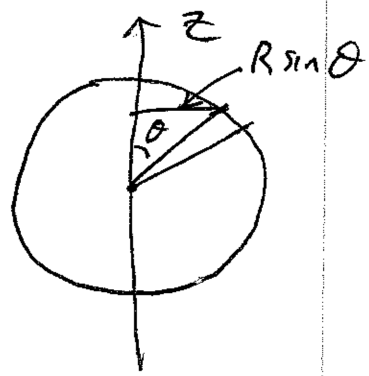


Ex: A ring of mass, through a diameter:

$$I = \int r^2 dm = \int_0^{2\pi} (R\sin\theta)^2 (\rho R\, d\theta)$$

$$= \tfrac{1}{2} \underbrace{(2\pi R\rho)}_{M} R^2 \qquad \left(\text{use } \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \tfrac{1}{2} MR^2$$

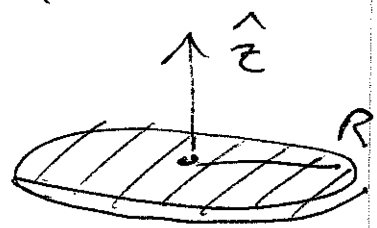


Ex: A disk of mass M and radius R, axis through the center; perpendicular to the plane:

$$I = \int r^2 dm = \int_0^{2\pi}\int_0^R r^2 (\rho\, r\, dr\, d\theta)$$

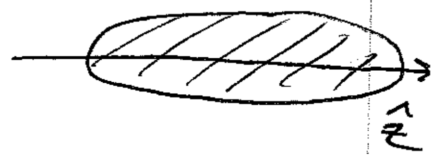
$$= \frac{R^4}{4}(2\pi\rho) = \tfrac{1}{2}\underbrace{(\rho\pi R^2)}_{M} R^2$$

$$= \tfrac{1}{2} MR^2$$



Ex: A disk of mass M and radius R, axis through center in the plane

Slice the disk into rings and use the previous result:



$$I = \int_0^R \underset{\substack{\uparrow \\ \text{from the previous result for a ring}}}{\tfrac{1}{2}}(\rho\, 2\pi r\, dr)\, r^2$$

$$= \frac{R^2}{4}(\rho\pi) = \frac{1}{4}\underbrace{(\rho\pi R^2)}_{M} R^2 = \frac{1}{4} M R^2 .$$

Generalized Angular Momentum and Rigid Body Motion.

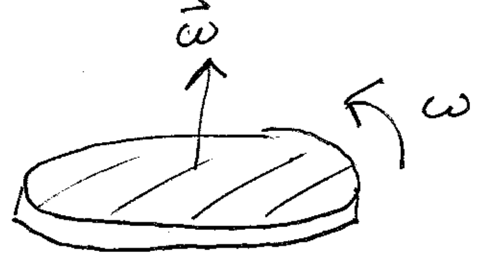
Now we release the conditions which make planar ~~motion~~ rotations simple. We will

- allow the body to have any shape in 3 dimensions
- allow it to rotate about any axis
- allow the axis of rotation to change (not fixed).

Preliminaries: The angular velocity vector $\vec{\omega}$.

$\vec{\omega}$ is a vector that describes the rotation axis and rotational velocity.

We'll define $\vec{\omega}$ to point along the axis of rotation, with its direction (+) or (−) determined by the right hand rule. Its magnitude will be ω, the instantaneous angular velocity.



$\vec{\omega}$ has some interesting properties:

1) Although it represents an angular <u>velocity</u>, $\vec{\omega}$ is <u>not</u> the time derivative of any <u>vector</u>. In other words, there is no "$\vec{\theta}$" vector such that $\boxed{\cancel{\vec{\omega} = \frac{d\vec{\theta}}{dt}}}$ (wrong !?!)

"$\vec{\theta}$" cannot be a vector <u>because rotations do not commute</u>. In other words,

$$"\vec{\theta}_1" + "\vec{\theta}_2" \neq "\vec{\theta}_2" + "\vec{\theta}_1"$$

You can easily confirm that rotating a book 90° about the x-axis, followed by a 90° rotation about the y-axis, <u>does not give the same result</u>. If "$\vec{\theta}$" were a true vector, then the order of addition would not matter.

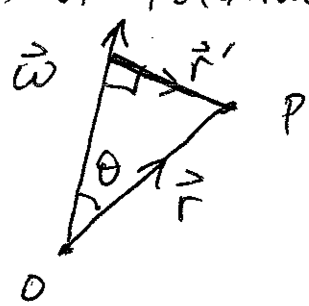
2) $\vec{\omega}$ does obey vector addition rules. For example, we can add together 2 rotations vectors to get a 3^{rd} legitimate rotation vector:

$$\vec{\omega}_3 = \vec{\omega}_1 + \vec{\omega}_2 \quad ✓$$

We can see this as follows. First,

> If an object is rotating with angular velocity $\vec{\omega}$, then the instantaneous velocity of a particular point in the body at location $\vec{r}$ is given by
> $$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

Proof: Drop a perpendicular from point P to the axis of rotation:



Then $\vec{v} = \vec{\omega} \times \vec{r}$ is orthogonal to $\vec{\omega}$, $\vec{r}$, and also $\vec{r}'$. So the direction of $\vec{v}$ is correct. (into the page)

Is the magnitude correct?

$$|\vec{v}| = |\vec{\omega}||\vec{r}|\sin\theta = \omega r' \quad ✓ \quad \text{yes, correct magnitude.}$$

Now lets add two rotation-vectors together:

Let coordinate systems S_1, S_2, and S_3 have a common origin. Let S_1 rotate with angular velocity $\vec{\omega}_{1,2}$ with respect to S_2, and let S_2 rotate with angular velocity $\vec{\omega}_{2,3}$ with respect to S_3. Question: does S_1 rotate instantaneously with velocity $\vec{\omega}_{1,3} \overset{??}{=} \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$

Answer: If $\vec{\omega}_{12}$ & $\vec{\omega}_{23}$ are colinear, then the result is clearly true, because the angular velocities add. But what if $\vec{\omega}_{12}$ & $\vec{\omega}_{23}$ point in different directions?

Pick P_1 at rest in S_1, and $\vec{r}$ is its position vector. Then its velocity in S_2 is

$$\vec{v}_{P1} = \vec{\omega}_{1,2} \times \vec{r}$$

Let P_2 be instantaneously identical to P_1, except P_2 is at rest in S_2. Then P_2's velocity is

$$\vec{v}_{P2} = \vec{\omega}_{2,3} \times \vec{r} .$$

Therefore, by addition of velocities, the velocity of P_1 relative to P_3 is

$$\vec{v}_{P1} + \vec{v}_{P2} = \underbrace{(\vec{\omega}_{1,2} + \vec{\omega}_{2,3})}\times\vec{r}$$

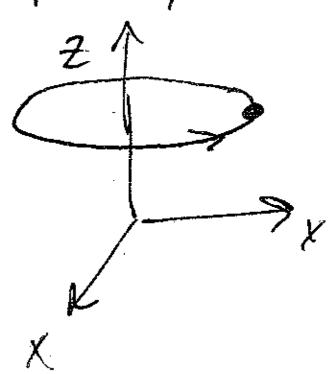
so the effective value of the vector $\vec{\omega}_{1,3}$ is just $\vec{\omega}_{12} + \vec{\omega}_{23}$.

3) Interesting property #3: Rotational velocities can only be represented by a vector in 3 dimensions. In 1D, there is no rotation. In 2D, rotational velocities are a simple number. In 3D, it takes 3 quantities to specify the rotation. In 4D, rotations can take place in 6 planes, so a 4-vector would not be sufficient.

Coordinate Systems: Note that a rotation must be specified with respect to a coordinate system. Specifying a single point (like the origin) or even an axis is not sufficient. For example, for a uniformly rotating body, we could choose to use a coordinate system which rotates with the body. In this case $\vec{\omega} = \emptyset$ by definition.

Kinematics of Rotations in 3D

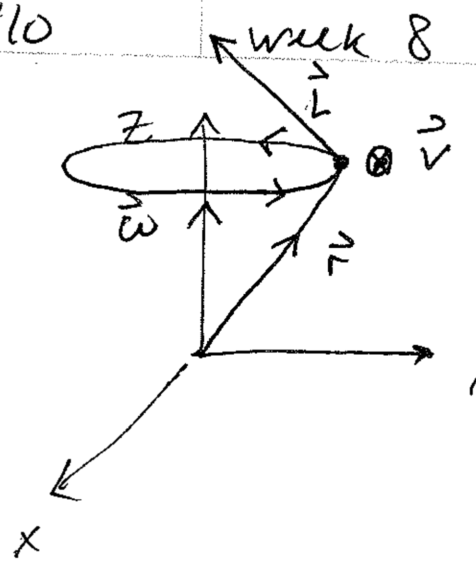
Start by considering a single particle traveling in a circular path, but not in the xy plane:



since it orbits the z-axis,

$$\vec{\omega} = (\emptyset, \emptyset, \omega_3)$$

What about $\vec{L}$?



Since $\vec{L} = \vec{r} \times m\vec{v}$, $\vec{L}$ is $\perp$ to $\vec{r}$ & $\vec{p}$

So $\vec{L}$ has both ~~as~~ a z component and a y component.

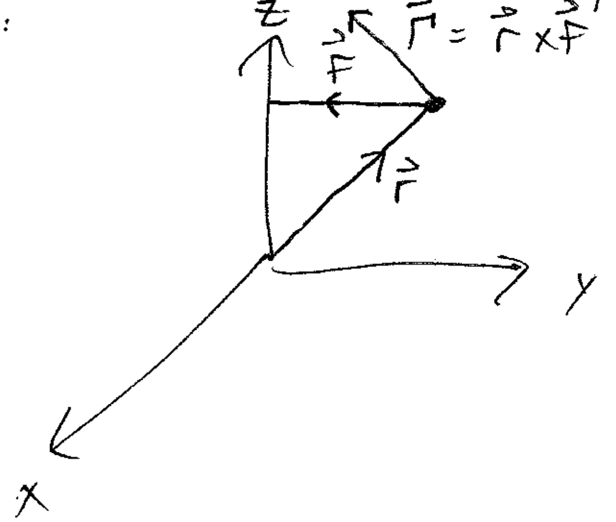
In other words, $\vec{L}$ & $\vec{\omega}$ are <u>not</u> colinear.

$\boxed{\vec{\omega} \text{ \& } \vec{L} \text{ are generally not colinear.}}$

As time goes forward, $\vec{L}$ changes direction, rotating about a cone. The change in $\vec{L}$ is caused by a torque:

$$\vec{\Gamma} = \frac{d\vec{L}}{dt}.$$

In this case the torque is due to the force which maintains the particle in its orbit:



Why should we expect that $\vec{L}$ has a y-component? Well, at the instant shown the velocity is into the page ($-\hat{x}$ direction). An observer who sees only this snapshot in time would say that there is angular momentum about both the y-axis and z-axis, but not the x-axis (because $\vec{v}$ is parallel to $-\hat{x}$.)

So $\vec{L} = (\emptyset, L_y, L_z)$ while $\vec{\omega} = (\emptyset, \emptyset, \omega_3)$.

It's clear that the relationship between $\vec{L}$ & $\vec{\omega}$ is not a simple proportionality constant.

$$\vec{L} \neq \alpha \vec{\omega}$$

In general we will need a matrix (actually a tensor) to describe how the various components of $\vec{\omega}$ get transformed into the components of $\vec{L}$:

$$\vec{L} = I\vec{\omega}$$

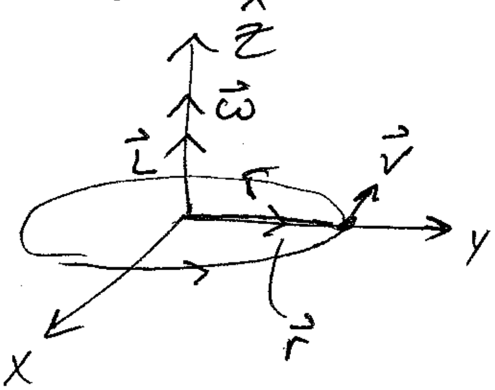
↑ a matrix (tensor), 3×3

or

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

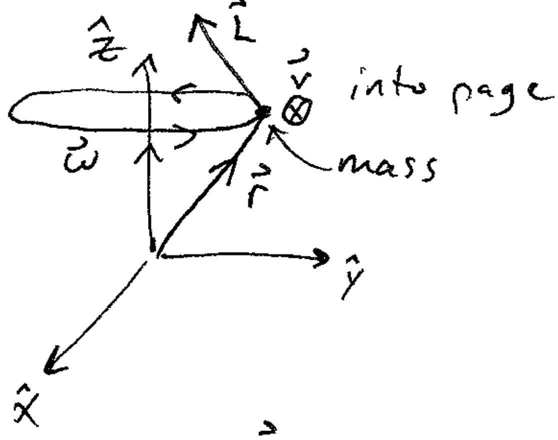
By dimensional analysis we can see that the elements of I must have units of $(\text{mass}) \times (\text{distance})^2$, the same as the moment of inertia. We call I the "inertia tensor"

Note #1 Note that the inertia tensor will depend on the choice of coordinate system. For example, if we choose to have the single particle orbit in the xy plane, then $\vec{L}$ will, in fact, be parallel to $\vec{\omega}$:



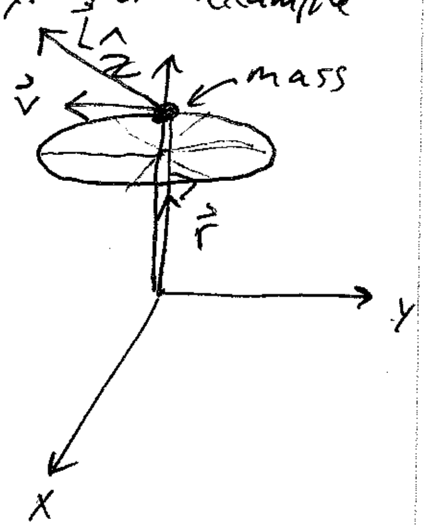
Now $\vec{r} \times m\vec{v} = \vec{L}$ is exactly in the $\hat{z}$ direction, so for this choice of origin, $\vec{L} = (\emptyset, \emptyset, L_z)$, colinear with $\vec{\omega}$. Therefore the inertia tensor must be different for this origin than for the displaced origin.

Note # 2 : Also note that the inertia tensor in general will depend on time, unless we rotate our coordinate system to remain fixed with respect to the rotating body. For example



time $t = \emptyset$, $\vec{L} = (\emptyset, L_y, L_z)$

while $\vec{\omega} = (\emptyset, \emptyset, \omega_3)$



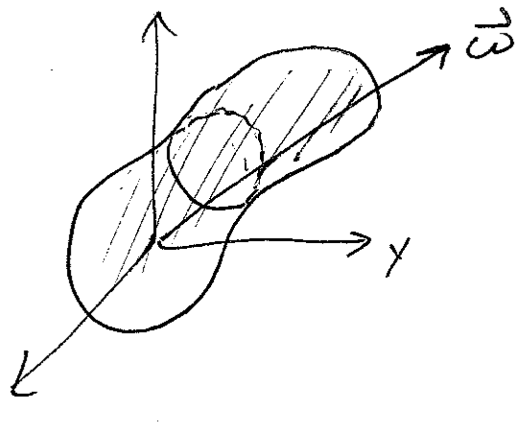
At a later time, $\vec{L} = (L_x, \emptyset, L_z)$ while $\vec{\omega} = (\emptyset, \emptyset, \omega_3)$.

Clearly I must change between these two moments in time, since $\vec{\omega}$ is the same, but $\vec{L}$ is different (and $\vec{L} = I\vec{\omega}$).

We will often choose to calculate I with respect to axes which, in effect, rotate with the body, so that I will appear constant (somewhat artificially).

The Inertia Tensor

It is straightforward to obtain an explicit expression for the inertia tensor. Consider an arbitrary body rotating about an arbitrary axis:



The total angular momentum is

$$\vec{L} = \int \vec{r} \times \vec{v}\, dm \quad , \text{ but } \vec{v} = \vec{\omega} \times \vec{r}, \text{ so}$$

$$\vec{L} = \int \vec{r} \times (\vec{\omega} \times \vec{r})\, dm$$

Simplify: $\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$

$$\begin{aligned} = & (\omega_2 z - \omega_3 y)\hat{x} \\ & + (-\omega_1 z + \omega_3 x)\hat{y} \\ & + (\omega_1 y - \omega_2 x)\hat{z} \end{aligned}$$

Then

$$\vec{r}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ (\omega_2 z-\omega_3 y) & (\omega_3 x-\omega_1 z) & (\omega_1 y-\omega_2 x) \end{vmatrix}$$

$$= \left(\omega_1(y^2+z^2) - \omega_2 xy - \omega_3 zx\right)\hat{x}$$
$$+ \left(\omega_2(z^2+x^2) - \omega_3 yz - \omega_1 xy\right)\hat{y}$$
$$+ \left(\omega_3(x^2+y^2) - \omega_1 zx - \omega_2 yz\right)\hat{z}$$

Therefore the angular momentum can be written

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} \int(y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ \int -xy\,dm & \int(z^2+x^2)\,dm & -\int yz\,dm \\ \int -zx\,dm & -\int yz\,dm & \int(x^2+y^2)\,dm \end{pmatrix}}_{\text{Inertia Tensor.}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

or $\boxed{\vec{L} = I\vec{\omega}}$

Simplest Case: A pancake object rotating in the xy plane.

Then $z=\emptyset$ ~~for all~~ for all dm. Then

$$I = \begin{pmatrix} \int y^2\,dm & -\int xy\,dm & \emptyset \\ -\int xy\,dm & \int x^2\,dm & \emptyset \\ \emptyset & \emptyset & \int(x^2+y^2)\,dm \end{pmatrix}$$

For the case that $\vec{\omega}$ is aligned on the $\hat{z}$ axis, we have $\vec{\omega} = (0, 0, \omega_3)$ and

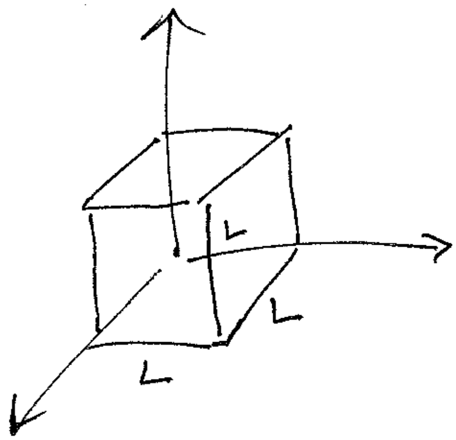
$$\vec{L} = \left(\int (x^2+y^2)\, dm \right) \omega_3 \, \hat{z}$$

or $L_z = I\omega_3$, with $I = \int (x^2+y^2)\,dm$ as before.

Ex: ~~Moment of Inertia~~ Inertia Tensor for a solid, uniform cube.

~~Due to the~~

We'll calculate with respect to an origin which is at one of the corners of the cube.



- First, note that due to the symmetry of the cube, we have

$$\int (y^2+z^2)\,dm = \int (x^2+z^2)\,dm = \int (x^2+y^2)\,dm$$

So all the diagonal elements are equal, and we need only calculate one of them. Let's calculate $I_{xx} = \int (y^2+z^2)\,dm$

$$dm = \rho\,(dx\,dy\,dz)$$

density $\uparrow$

So $I_{xx} = \int_0^L \int_0^L \int_0^L (y^2+z^2)\,\rho\, dx\,dy\,dz$

$$= \rho\int_0^L dx\int_0^L dz\int_0^L y^2 dy + \rho\int_0^L dx\int_0^L dy\int_0^L z^2 dz$$

$$= \rho L^2\left[\tfrac{1}{3}L^3 + \tfrac{1}{3}L^3\right]$$

$$= \tfrac{2}{3}\underbrace{(\rho L^3)}_{M = \text{total mass}} L^2$$

$$\boxed{I_{xx} = \tfrac{2}{3}ML^2 = I_{yy} = I_{zz}}$$

• Off-diagonal elements: By symmetry,

$$-\int xy\,dm = -\int xz\,dm = -\int yz\,dm,$$

so we need only calculate one of these:

$$I_{xy} = -\int_0^L\int_0^L\int_0^L xy\,\cancel{dm}(\rho\,dx\,dy\,dz)$$

$$= -\rho L\int_0^L\int_0^L xy\,dx\,dy$$

$$= -\rho L\int_0^L \tfrac{1}{2}L^2 x\,dx$$

$$= -\tfrac{1}{2}\rho L^3\left(\tfrac{1}{2}L^2\right)$$

$$= -\tfrac{1}{4}\underbrace{(\rho L^3)}_{M} L^2$$

$$\boxed{I_{xy} = -\tfrac{1}{4}ML^2 = I_{xz} = I_{yz}}$$

Therefore

$$I = ML^2\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} = \frac{ML^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Now we can calculate $\vec{L}$ for some specific $\vec{\omega}$:

For example, suppose the cube rotates around the x-axis: $\vec{\omega} = (\omega_1, 0, 0)$, then

$$\vec{L} = I\vec{\omega} = \frac{ML^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}\begin{pmatrix} \omega_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{L} = \frac{ML^2\omega_1}{12}\begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$

At this moment, the cube has angular momentum about $+x$, $-y$, and $-z$.

Note that as the cube rotates, the inertia matrix will change, unless we rotate the axes of our coordinate system along with the cube.

Kinetic Energy: Can be written as

$$T = \int \tfrac{1}{2} v^2\, dm = \int \tfrac{1}{2} |\vec{\omega}\times\vec{r}|^2\, dm$$

$$= \tfrac{1}{2}\int\left[(\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2\right] dm$$

By multiplying this out, we can show that

$$T = \frac{1}{2}(\omega_1, \omega_2, \omega_3)\begin{pmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\boxed{T = \frac{1}{2}\vec{\omega}\, I\, \vec{\omega}}$$

This is the correct generalization of $T = \frac{1}{2} I_z \omega^2$ in the simple fixed-axis planar case.

Also, since $\vec{L} = I\vec{\omega}$, we can also write

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

Parallel Axis Theorem

The correct generalization (which we won't prove here) is as follows:

If we know the inertia tensor computed about the center of mass, then we can find the inertia tensor for any other origin by calculating

$$I = I_R + I_{cm},$$

where $I_R = M \begin{pmatrix} Y^2+Z^2 & -XY & -XZ \\ -XY & X^2+Z^2 & -YZ \\ -XZ & -YZ & X^2+Y^2 \end{pmatrix}$

and the X, Y, Z are the coordinates of the center of mass w/respect to the new origin: $\vec{R} = (X, Y, Z)$.

Note that "Parallel Axis Theorem" is a misnomer for this result, because the inertia tensor tells us about 3 axes at once. ⇒ It only depends on the choice of origin, not on a specific rotation direction.

Principle Axes

Notice that the Inertia Tensor is a real, symmetric matrix.

↳ Elements across the diagonal are the same.

For example,

$$I_{xy} = -\int xy\,dm = -\int yx\,dm = I_{yx}$$

$$\Rightarrow \boxed{I_{xy} = I_{yx}}$$

Notice that this means that there are only <u>six</u> independent numbers in I (three moments of inertia: I_{xx}, I_{yy}, I_{zz}) (and three "products of inertia": I_{xy}, I_{yz}, I_{xz}).

Now we can invoke a nice theorem from linear algebra which says the following:

> Given a real symmetric 3x3 matrix I, there exists 3 orthonormal, real vectors, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, and 3 real numbers λ_1, λ_2, and λ_3, such that
> $$I\hat{e}_1 = \lambda_1 \hat{e}_1$$
> $$I\hat{e}_2 = \lambda_2 \hat{e}_2$$
> $$I\hat{e}_3 = \lambda_3 \hat{e}_3 .$$

In other words, I has three eigenvectors and 3 eigenvalues, and the eigenvectors are orthonormal.

Physically, what this means is that if we rotate a body around one of the eigenvectors, then the ~~ang~~ angular momentum vector $\vec{L}$ will be parallel to $\vec{\omega}$:

Let $\vec{\omega} = \omega\hat{e}_1$. Then $\vec{L} = I\vec{\omega} = I\omega\hat{e}_1$

$$= \omega(I\hat{e}_1)$$

$$\vec{L} = \omega\lambda_1\hat{e}_1$$

In other words, $\vec{L}$ is parallel to $\vec{\omega}$ if $\vec{\omega}$ points along an eigenvector of I.

We call the eigenvectors of I the "principle axes". Usually our first goal after encountering any inertia tensor is to find the principle axes. Then, <u>if we choose the principle axes as our coordinate system, I will appear very simple</u>:

$$I = \underbrace{\begin{pmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{pmatrix}}_{}$$

in the coordinate system of $\hat{e}_1, \hat{e}_2, \hat{e}_3$.

Furthermore, if an isolated object is suspended in space and rotating about a principle axis, <u>it will not wobble</u>. This follows because $\vec{L}$ points along $\vec{\omega}$, and $\vec{L}$ will be constant for an isolated object. Therefore $\vec{\omega}$ will be constant.

In the general case, where the object rotates around some arbitrary axis, $\vec{L}$ must still be constant, but $\vec{\omega}$ will not be parallel. Therefore $\vec{\omega}$ ~~will~~ will move around, and the object wobbles.

It is important to note that any object, no matter how lopsided, potato-shaped, or ugly, will have 3 principle axes about which it will rotate without wobbling. This result is not physically obvious, but the linear algebra guarrantees that it is true.

Finding the Principle Axes.

This is a classic eigenvalue problem. We seek vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ such that

$$I(\omega \hat{e}_1) = \omega(I\hat{e}_1) = \omega\lambda\hat{e}_1$$

Or simply let $\vec{\omega}$ be our angular velocity vector. We seek $\vec{\omega}$ such that

$$I\vec{\omega} = \lambda\vec{\omega}$$

or $(I - \lambda \mathbb{1})\vec{\omega} = \emptyset$

or $\det(I - \lambda \mathbb{1}) = \emptyset$ ⇐ characteristic equation

Ex: Cube, rotating about a corner.

$$I = \mu \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}, \text{ when } \mu = \frac{ML^2}{12}$$

Find the eigenvalues:

$$\det \begin{pmatrix} 8\mu-\lambda & -3\mu & -3\mu \\ -3\mu & 8\mu-\lambda & -3\mu \\ -3\mu & -3\mu & 8\mu-\lambda \end{pmatrix} = \emptyset$$

$$(8\mu-\lambda)\left[(8\mu-\lambda)^2 - (-3\mu)^2\right]$$
$$+ 3\mu\left[(-3\mu)(8\mu-\lambda) - (-3\mu)^2\right]$$
$$- 3\mu\left[(-3\mu)^2 - (-3\mu)(8\mu-\lambda)\right]$$
$$= (2\mu-\lambda)(11\mu-\lambda)^2 \qquad \text{simplify}$$

The three roots are $\lambda = 2\mu,\ 11\mu,$ and 11μ

Now find the eigenvalues:

$\lambda = 2\mu$: $\quad (I - \lambda 1)\omega = \emptyset$

$$\mu \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \emptyset$$

$$6\omega_1 - 3\omega_2 - 3\omega_3 = \emptyset \qquad (1)$$
$$-3\omega_1 + 6\omega_2 - 3\omega_3 = \emptyset \qquad (2)$$
$$-3\omega_1 - 3\omega_2 + 6\omega_3 = \emptyset \qquad (3)$$

Subtract ② from ①:

$$9\omega_1 - 9\omega_2 = 0 \Rightarrow \boxed{\omega_1 = \omega_2}$$

Then, from the ①, $6\omega_1 - 3\omega_1 - 3\omega_3 = 0$

$$3\omega_1 - 3\omega_3 = 0$$

$$\boxed{\omega_1 = \omega_3}$$

$$\Rightarrow \vec{\omega} = (\omega_1, \omega_1, \omega_1)$$

or $\boxed{\hat{e}_1 = \text{unit vector} = \frac{1}{\sqrt{3}}(1,1,1) \quad \text{for } \lambda = 2\mu}$

⇒ The principal diagonal of a cube is a principal axis for rotation through a corner.

other case:

$\lambda = 11\mu$:
$$\mu \begin{pmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0$$

$\omega_1 + \omega_2 + \omega_3 = 0$ ⇐ defines a plane

or $(\omega_1, \omega_2, \omega_3) \cdot (1,1,1) = 0$

$\Rightarrow$ Any vector orthogonal to $(1,1,1)$ is a principal axis.

We can pick any 2, for example

$$\hat{e}_2 = \frac{1}{\sqrt{2}}(\emptyset, 1, -1) \quad \text{for } \lambda = 11\mu$$

$$\text{and } \hat{e}_3 = \frac{1}{\sqrt{6}}(2, -1, -1) \quad \text{for } \lambda = 11\mu$$

Now if we had chosen these directions as the x, y, z axes of our coordinate system, we would have found, when computing the inertia tensor,

$$\underline{I} = \begin{pmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{pmatrix}$$