Differential Equations

- $\frac{dy}{dt} = A \Rightarrow y(t) = At + C$
- $\frac{dy}{dt} = Ay \Rightarrow y(t) = Ce^{At}$
- $\frac{d^2y}{dt^2} + Ay = 0 \Rightarrow y(t) = c_1 \sin(\sqrt{A}t + c_2)$ (and other way to write it.)

Euler Method: (Numerical)

$$v_{i+1} = v_i + a_{i+1}\Delta t$$

$$x_{i+1} = x_i + v_{i+1}\Delta t$$

$$a_{i+1} = \text{equation of motion}$$

1 Dimensional Single Particle Mechanics:

IF $F = F(t)$, then $v(t) = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t')dt'$

$$x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'$$

IF $F = F(x)$, then use $\cancel{a = \frac{dv}{dt}}$ $a = v\frac{dv}{dx}$, or

$$v(x) = \left[v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')dx'\right]^{\frac{1}{2}}$$

and $t = t_0 + \int_{x_0}^{x(t)} \frac{dx'}{v(x')}$

IF $F = F(v)$, then

$t = t_0 + m\int_{v_0}^{v(t)} \frac{dv'}{F(v')}$, then solve for $v(t)$. and integrate to get $x(t)$.

Newtons laws:

2) $\vec{F} = m\vec{a} = m\ddot{\vec{r}} = \dot{\vec{p}}$

3) $\vec{F}_{12} = -\vec{F}_{21}$ ⇒ leads to conservation of momentum

Polar coordinates

$$\vec{r} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r}$$, and so

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

Air resistance

$$\vec{F}_v = -b\vec{v}$$ (linear drag)

or $\vec{F}_v = -cv^2\hat{v}$ or $\vec{F}_v = -cv\vec{v}$

$$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x$$

$$m\dot{v}_y = mg - c\sqrt{v_x^2 + v_y^2}\, v_y$$

↑ with gravity

Charged particle in uniform field: $\omega = qB/m$

$$\dot{v}_x = \omega v_y$$
$$\dot{v}_y = -\omega v_x$$

⇒ $\xi = x + iy = \sim e^{-i\omega t}$

while $z(t) = z_0 + v_{0z}t$

Conservation of Momentum

$$\dot{\vec{P}}_{total} = \vec{F}_{external} \Rightarrow \vec{P}_{total} = \text{constant}$$

when $\vec{F}_{external} = 0$.

Rocket Motion: $\boxed{m(t)\dot{v} = -\dot{m}(t)v_{ex}}$

Center of Mass: $M \equiv \sum_\alpha m_\alpha$ = total mass

$$\vec{R}_{CM} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{1}{M}\int_{\text{volume}} \rho\, \vec{r}\, dV$$

Angular Momentum: (single particle)

$$\vec{\ell} = \vec{r} \times \vec{p}$$

$$\dot{\vec{\ell}} = \vec{r} \times \vec{F} \equiv \vec{\Gamma}$$

$\boxed{\dot{\vec{\ell}} = \vec{\Gamma}}$ Newton's 2nd Law in angular form.

Total Angular Momentum of a system of particles:

$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha$. Then $\dot{\vec{L}} = \vec{\Gamma}^{ext}$.

Moment of Inertia

$$I \equiv \sum_\alpha m_\alpha \underset{\substack{\uparrow \\ \text{distance} \\ \text{to rotation} \\ \text{axis}}}{\rho_\alpha^2} = \int \underset{\substack{\uparrow \\ \text{density} \\ \text{function}}}{\rho(x,y,z)} \underset{\substack{\uparrow \\ \text{distance to axis.}}}{\rho_\alpha^2}\, dV$$

For a fixed rotation axis, $L_z = I\omega$

$$\text{or} \quad I\dot{\omega} = \Gamma_z$$

Work & Energy

$$dT = \vec{F}\cdot dr \quad \Rightarrow \quad \Delta T = \int_1^2 \vec{F}\cdot d\vec{r}$$

If $\vec{\nabla}\times\vec{F} = 0$, then the work of $\vec{F}$ is path independent, and we say it is conservative. We can find a function U such that

$$U(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

Then $\Delta T + \Delta U = \Delta(T+U) \equiv \Delta E = 0$
(energy is conserved.)

In One-Dimension

$$W_{12} = \int_{x_1}^{x_2} F(x')\,dx' \,, \quad U(x) = -\int_{x_0}^{x} F(x')\,dx' \,,$$

$F_x = -\frac{dU}{dx}$. For a conservative system,

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E-U(x)} \quad \text{or}$$

$$\boxed{t_F = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-U(x')}}}$$

Small oscillations

Near a local minimum in the potential,

$$U(x_0+a) \approx U(x_0) + U'(x_0)a + \tfrac{1}{2}U''(x_0)a^2 + \dots$$

Then $\omega_0 \approx \sqrt{\dfrac{U''(x_0)}{m}}$

Damped Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \Rightarrow x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$
$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

For $\beta < \omega_0$, define $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, so

$$x(t) = A e^{-\beta t}\cos(\omega_1 t - \delta)$$

initial conditions

Driving Forces

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \equiv f(t)$$

The soln is

$$x(t) = \underset{\substack{\uparrow \\ \text{no free} \\ \text{parameter}}}{x_{\text{particular}}(t)} + \underset{\substack{\uparrow \\ \text{2 free} \\ \text{parameters}}}{x_{\text{transient}}(t)}$$

For $F(t) = f_0 \cos(\omega t)$,

$$x_{particular}(t) = A\cos(\omega t - \delta),$$

$$A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$

Periodic Driving Forces

Let $$F(t) = \sum_{n=0}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

where $\omega = \frac{2\pi}{T}$ and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos(n\omega t)\,dt \quad , n \geq 1$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega t)\,dt \quad , n \geq 1$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} F(t)\,dt$$

If $F(t)$ has cosine terms only, then

$$x(t) = \sum_n x_n(t) \quad , \quad x_n(t) = A_n \cos(n\omega t - \delta_n),$$

$$A_n = \frac{a_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2}}$$

and $$\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$

Green's Method for a damped oscillator, arbitrary forcing function.

For a forced, damped oscillator,

$$x(t) = \int_{-\infty}^{t} F(t')\, G(t,t')\, dt' \;,$$ where

$$G(t,t') = \begin{cases} \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t')) \,, & \text{for } t \geq t' \\ \phi \,, & \text{for } t < t' \end{cases}$$

Lagrangian Mechanics

$\mathcal{L} = T - U$. Then

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

If q does not appear in the Lagrangian, then we say that q is "ignorable" or "cyclic"

This leads to a conservation law.