Exam Review

~~The Action~~

Hamilton's principle

The "action" is $S \equiv \int_{t_1}^{t_2} \mathcal{L}(\dot{x}, x, t)\, dt$, $\mathcal{L}$ = Lagrangian

We examine small variations around the true path

$$\underbrace{\mathbb{X}(t)}_{} = \underbrace{x(t)}_{\text{true path}} + \underbrace{\alpha}_{\text{small parameter}} \underbrace{\eta(t)}_{\text{variation}}$$

We say the action is stationary if S has no first-order dependence on the small parameter α. Hamilton's Principle says that the action should be stationary: $\delta S = 0$

This leads to the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \quad \text{for } x$$

or

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad \text{for a generalized coordinate } q_i.$$

Calculus of Variations

Any mathematical problem which can be formulated as ~~the~~ finding ~~an~~ maximum or minimum value for an integral of the form

$$\int_{x_1}^{x_2} F\left[y(x), y'(x), x\right] dx$$

is solved by the Euler Lagrange equation

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad \text{where } y' = \frac{dy}{dx}$$

For example, the length of a curve $y(x)$ in the xy plane is given by

$$\int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2}\, dx$$

To find the minimum length between x_1 & x_2 we solved $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ for $F = \sqrt{1+(y')^2}$.

<u>Central Force Motion</u>

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \text{let } \vec{r} \equiv \vec{r}_1 - \vec{r}_2$$

$$T = \frac{1}{2}\left(M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right), \quad \mu = \frac{m_1 m_2}{m_1+m_2} = \text{"reduced mass"}$$

$$\mathcal{L} = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{cm}} + \underbrace{\left(\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)}_{\mathcal{L}_{relative}}$$

Use Polar Coordinates for the relative motion, with the origin at the CM:

$$\mathcal{L}_{rel} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

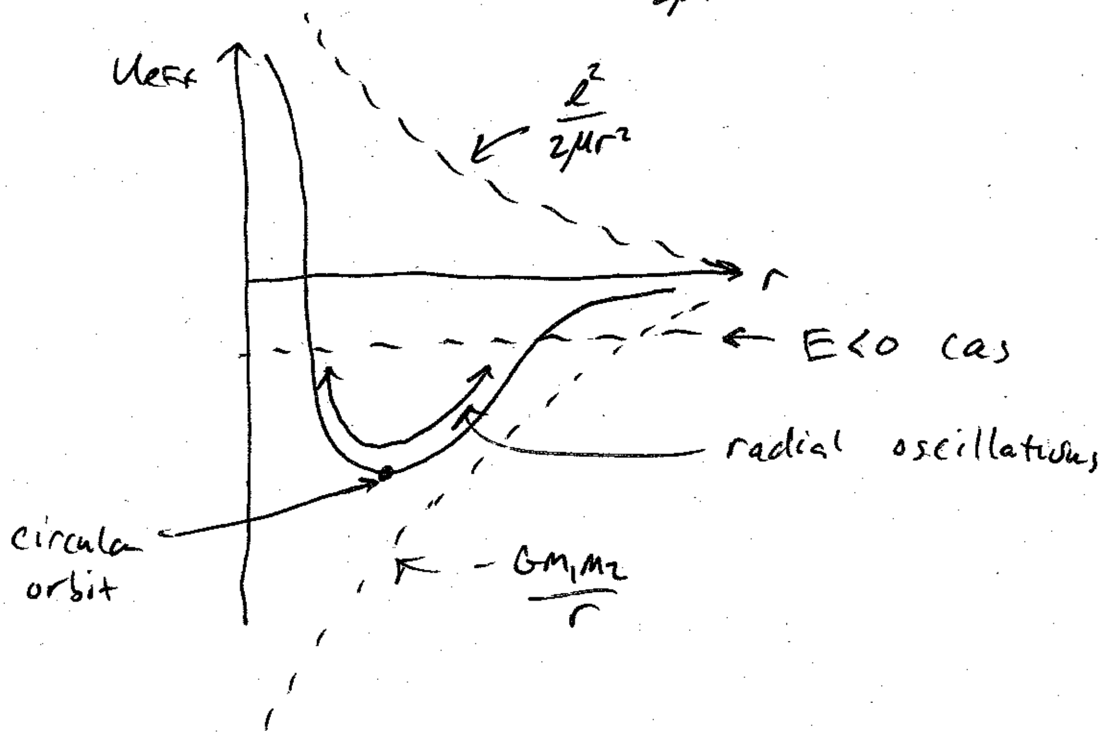
$$\Rightarrow \dot{\phi} = \frac{\ell}{\mu r^2}, \quad \ell = \text{constant} \quad (\text{angular momentum})$$

Radial Equation:

$$\mu\ddot{r} = -\frac{d}{dr}\underbrace{\left[U(r) + \frac{\ell^2}{2\mu r^2}\right]}_{U_{eff}(r)}$$

Energy conservation: $\tfrac{1}{2}\mu\dot{r}^2 + U_{eff} = \text{constant} = E.$

For gravity, $U_{eff} = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$



Orbital Equation: (determining the shape of the orbit)

$$u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u^2(\phi)} F \quad \text{where } u \equiv \frac{1}{r}$$

For the case of the inverse square law,

$$F(r) = -\frac{\gamma}{r^2} = -\gamma u^2 \quad , \gamma = G m_1 m_2$$

Then the orbital equation is solved by

$$r(\phi) = \frac{c}{1+\epsilon \cos\phi}$$

where $c = \frac{\ell^2}{\gamma\mu}$ = units of length

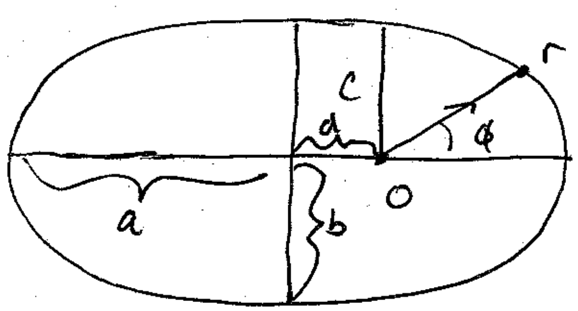
ϵ = eccentricity : $\epsilon = 0$ (circular orbit)

$0 < \epsilon < 1$ (elliptical)

$\epsilon = 1$ (parabolic)

$\epsilon > 1$ (hyperbolic)

For an elliptical orbit,



$$a = \frac{c}{1-\epsilon^2}$$

$$b = \frac{c}{\sqrt{1-\epsilon^2}}$$

$$d = a\epsilon$$

Also, $\epsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$

Kepler's Third Law: $\tau^2 = \frac{4\pi^2 \mu a^3}{\gamma}$ ← semi-major axis length

↑ τ = period

Relationship between energy and ε and l:

$$E = \frac{\gamma^2 \mu}{2l^2}(\varepsilon^2 - 1)$$

Angular change: $\phi(r) = \int_{r_1}^{r_2} \frac{\frac{l}{r^2}\,dr}{2\mu\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}$

Angular Momentum and Rigid Bodies

For a fixed rotation axis, and ignoring the components of $\vec{L}$ that are $\perp$ to the rotation axis, we have

$$L_z = I_z \omega \quad , \quad I_z = \int (x^2 + y^2)\,dm = \int r^2\,dm = \sum_i m_i r_i^2$$

Then $T = \frac{1}{2} I_z \omega^2$

Including the motion of the CM,

$$\vec{L} = \vec{R}_{cm} \times \vec{P} + \left(I_z^{cm} \omega'\right)\hat{z}$$

↑ rotational angular velocity about the center of mass.

If the rotation axis is not fixed, and/or we desire to know about all 3 components of $\vec{L}$, then we have

$$\vec{L} = I\vec{\omega}, \quad \vec{\omega} = \text{angular velocity vector}$$
$$I = \text{inertia tensor}$$

$$I = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)dm \end{pmatrix}$$

Kinetic Energy: $T = \frac{1}{2}\vec{\omega}\, I\, \vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

Principal Axes: We can always find at least 3 orthogonal directions in space (unit vectors) for which

$$I\hat{e}_1 = \lambda_1 \hat{e}_1 \qquad \hat{e}_1, \hat{e}_2, \hat{e}_3 \text{ are eigenvectors}$$
$$I\hat{e}_2 = \lambda_2 \hat{e}_2 \qquad \lambda_1, \lambda_2, \lambda_3 \text{ are the eigenvalues}$$
$$I\hat{e}_3 = \lambda_3 \hat{e}_3$$

We call the eigenvectors the principal axes, and the $\{\lambda\}$, the principal moments.

If we choose the principal axes as the coordinate system, then I will appear as a diagonal matrix:

$$I = \begin{pmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{pmatrix}$$

In the system of the principal axes,

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

$\omega_1, \omega_2, \omega_3$ are the projections of $\vec{\omega}$ onto the principal axes at each moment.

If we project $\frac{d\vec{L}}{dt}$ and $\vec{\omega}$ onto the principal axes at each moment (even though the principle axes keep rotating), we have

$$\left(\frac{d\vec{L}}{dt}\right)_1 = \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2$$

$$\left(\frac{d\vec{L}}{dt}\right)_2 = \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3$$

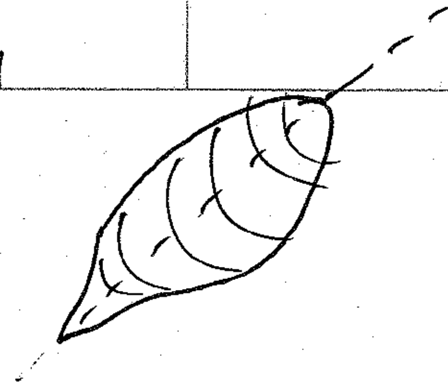
$$\left(\frac{d\vec{L}}{dt}\right)_3 = \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1$$

And since $\vec{\Gamma} = \frac{d\vec{L}}{dt}$ we have

$$\left.\begin{aligned} \Gamma_1 &= \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\omega_3\omega_2 \\ \Gamma_2 &= \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\omega_1\omega_3 \\ \Gamma_3 &= \lambda_3\dot{\omega}_3 + (\lambda_2-\lambda_1)\omega_2\omega_1 \end{aligned}\right\} \text{ Euler's Equations.}$$

These are most useful for the free precession of a top, when $\Gamma_1 = \Gamma_2 = \Gamma_3 = \emptyset$.

Symmetric Free Top:

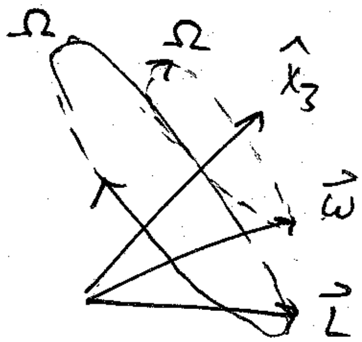


$$I = \begin{pmatrix} I & \emptyset & \emptyset \\ \emptyset & I & \emptyset \\ \emptyset & \emptyset & I_3 \end{pmatrix}$$

Then $\vec{\omega} = \underbrace{(\omega_0 \cos(\Omega t), -\omega_0 \sin(\Omega t), \omega_3)}$

projected onto the rotating body axes

where $\Omega = \left(\frac{I - I_3}{I}\right)\omega_3$



$\hat{x}_3$, $\vec{\omega}$, and $\vec{L}$ remain coplanar. $\vec{\omega}$ & $\vec{L}$ precess about $\hat{x}_3$ as viewed from the body frame

Mechanics in Non-inertial Frames of reference.

This topic is important primarily because the earth's surface is a non-inertial frame of reference (due to the rotation of The earth.)

We can ~~do~~ analyze a mechanical system from the point of view of a non-inertial frame of reference, as long as we add the necessary pseudo-forces that make Newton's 2nd Law still hold. (effectively).

Let S_0 be an inertial frame and S be a frame that is accelerating with respect to S_0.

In S_0, we have $m\ddot{\vec{r}}_0 = \vec{F}$

In S, we have $\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$ ← velocity of S relative to S_0

velocity in S

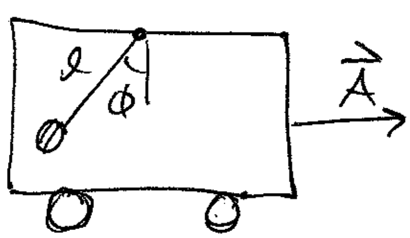
and $\ddot{\vec{r}} = \ddot{\vec{r}}_0 - \vec{A}$ ← acceleration of S relative to S_0

Therefore $m\ddot{\vec{r}} = \underbrace{m\ddot{\vec{r}}_0}_{\vec{F}} - m\vec{A}$

$$\boxed{m\ddot{\vec{r}} = \vec{F} - m\vec{A}}$$

So we can apply Newton's 2nd Law in the non-inertial frame, but we must add an additional force-like term $(-m\vec{A})$. This is called a "pseudoforce" or "fictitious force". ~~one~~ Pseudoforces are always proportional to the mass. Curiously, the gravitational force is also proportional to mass, which raises the question that gravity might be a pseudoforce. In fact, in general relativity, gravitational effects are essentially treated as an artifact of the choice of coordinate system, just like a pseudoforce.

Ex: Pendulum in an accelerating car

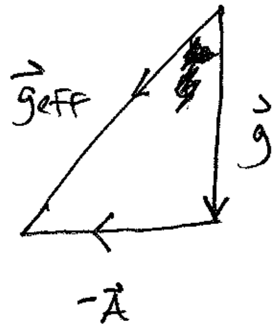


The forces are $\vec{T}$ (tension) and $m\vec{g}$.

From the perspective of a person inside the car, we also have a pseudoforce:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} - m\vec{A} = \vec{T} + m(\vec{g} - \vec{A})$$
$$= \vec{T} + m\vec{g}_{eff}$$

where $\vec{g}_{eff} = \vec{g} - \vec{A}$



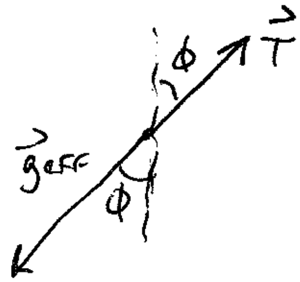
If the pendulum remains at rest (no oscillations), then

$$\vec{T} = -m\vec{g}_{eff}$$

And the direction of $\vec{T}$ tells us that

$$\phi_{equilibrium} = \tan^{-1}\left(\frac{A}{g}\right)$$

(because



The frequency of small oscillations is

$$\omega = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\sqrt{g^2 + A^2}}{L}}$$

Tides

The earth and moon revolve around their common center-of-mass. Therefore ~~every drop of water on~~ the earth ~~surface accelerates towards the common CM, which point~~. accelerates towards the moon. We take the center of the earth as our origin and add a pseudoforce to account for the earth's acceleration.

Each drop of water experiences theses forces:

1) $m\vec{g}$ of the earth

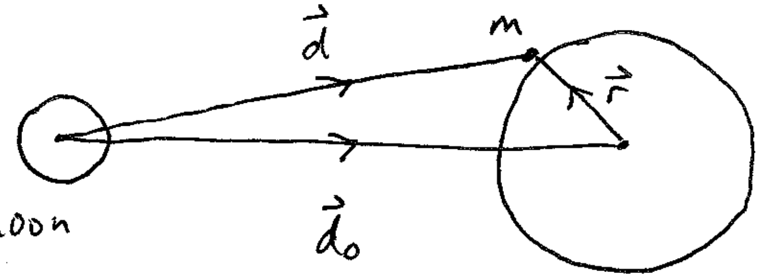
2) $-GM_m \frac{\hat{d}}{d^2}$, $\vec{d}$ points towards the moon, This is the gravitational force due to the moon.

3) $\vec{F}_{ng}$, a non-gravitational force such as the buoyant force. This holds the drop of water fixed in the earth's frame

4) A pseudoforce due to earth's acceleration

$$\vec{A} = -GM_m \frac{\hat{d}_0}{d_0^2}$$

, $\vec{d}_0$ is the position of earth's center relative to the moon.



The pseudo force is $-m\vec{A} = GM_m \frac{\hat{d}_0}{d_0^2}$

So we have

$$m\ddot{\vec{r}} = m\vec{g} - GM_m m \frac{\hat{d}}{d^2} + \vec{F}_{ng} + GM_m m \frac{\hat{d}_0}{d^2}$$

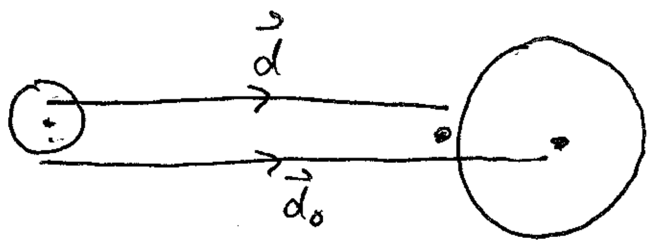
Define $\vec{F}_{tidal} \equiv -G M_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)$

Then $m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tid} + \vec{F}_{ng}$

In the absence of $\vec{F}_{tidal}$ this would be our normal equation of motion for any object on the earth surface (if the earth's surface were an inertial frame.)

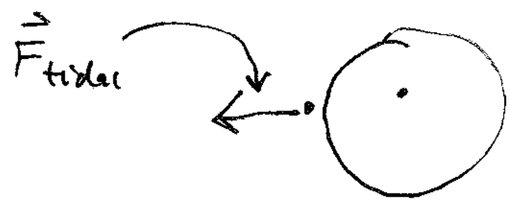
We can see the effect of the moon in $\vec{F}_{tidal}$

For a point near the moon:

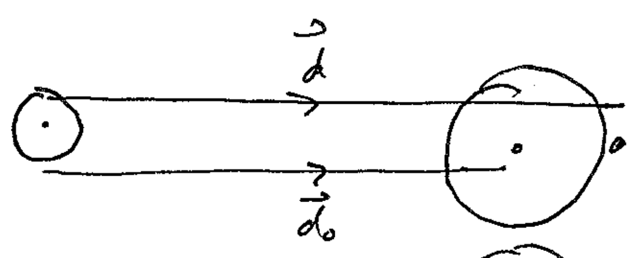


$\vec{d}$ is smaller than $\vec{d}_0$,

so



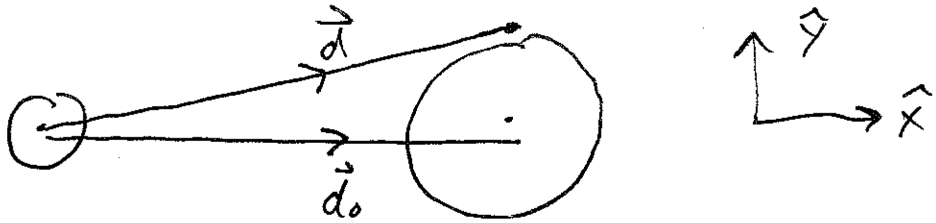
For a point opposite the moon:



$\vec{d}_0$ is smaller than $\vec{d}$,

so

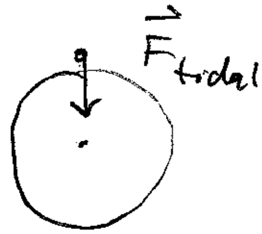
At 90° from the moon we have



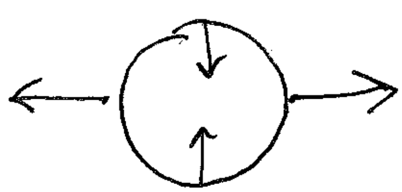
In $\vec{F}_{tidal\,y} = -GM_m m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)$

~~Bsz~~ $\frac{\hat{d}_0}{d_0^2}$ has only an $\hat{x}$ component

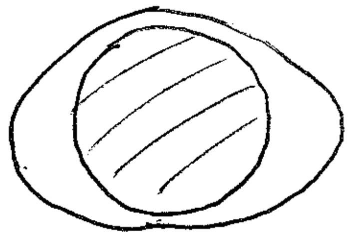
$\frac{\hat{d}}{d^2}$ has both x and y components. Because the moon is much further away than the radius of the earth, the x component of $\frac{\hat{d}}{d^2}$ will almost exactly cancel $\frac{\hat{d}_0}{d_0^2}$. This leaves only the $\hat{y}$ component:



So the total effect of $\vec{F}_{tidal}$ looks like



This gives the ocean 2 bulges.



and 2 ~~h~~ high tides per day as the earth rotates.

Rotating Frames of Reference.

For the case of a rotating frame, with angular velocity $\vec{\Omega}$, we have

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{V}$$

, $\vec{V}$ is velocity due to rotation of the frame

$= \vec{\Omega} \times \vec{r}$

so $$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega} \times \vec{r}$$

We can consider this equation to be an operator which acts on $\vec{r}$:

$$\left(\frac{d}{dt}\right)_{S_0} = \text{operator} = \left[\left(\frac{d}{dt}\right)_S + \vec{\Omega} \times\right]$$

To get an expression for the acceleration as viewed from the rotating frame, we can apply the operator twice to $\vec{r}$:

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$$

$$\left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

$$+\vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

Let's use the "dot" notation to describe time derivatives in the S system:

$$\dot{\vec{r}} = \left(\frac{d\vec{r}}{dt}\right)_S$$

Then

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \ddot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \frac{\vec{F}}{m}$$ according to Newton's 2^{nd} Law

so

$$\boxed{m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}}$$

where we have reversed the order of the cross products to get rid of minus signs.

The additional terms on the right hand side are pseudo forces. They have names:

$$\vec{F}_{\text{coriolis}} \equiv 2m\dot{\vec{r}}\times\vec{\Omega}$$

and $$\vec{F}_{\text{centrifugal}} = m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$

For objects on the earth's surface they have magnitudes

$$|\vec{F}_{cor}| \sim mv\Omega$$

and $$|\vec{F}_{CF}| \sim mr\Omega^2 \quad , \; r = R_{earth}$$

so $$\frac{|\vec{F}_{cor}|}{|\vec{F}_{CF}|} \sim \frac{v}{R\Omega} \sim \frac{v}{V}$$

rotational velocity of earth's surface.

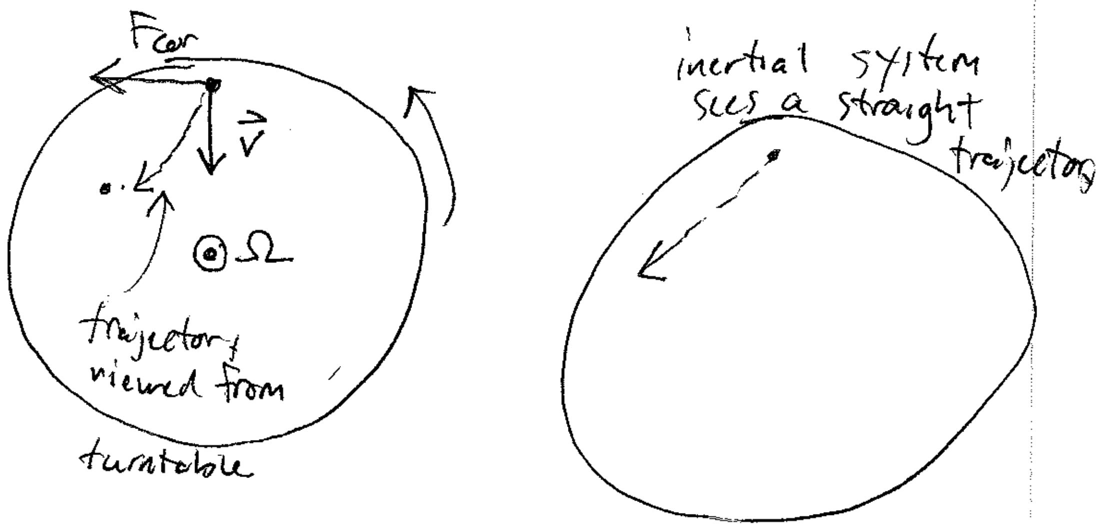
Earth rotational velocity at the surface (near the equator) is $\sim$ 1000 miles/hour. so if the velocity of the object is small compared to this then we can probably ignore the coriolis force and can consider only the centrifugal force.

Coriolis Force

$$\vec{F}_{cor} = 2m\vec{v} \times \vec{\Omega}$$

This can be pictured as a magnetic-like force, with $2m \to q$ and $\vec{\Omega} \to \vec{B}$

On a turntable,

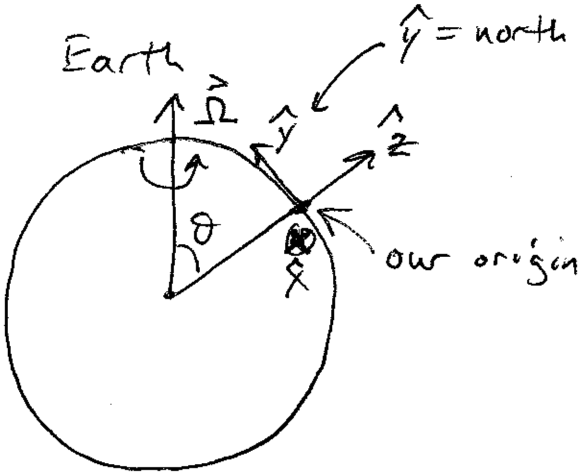


In the northern hemisphere, hurricanes rotate counterclockwise due to the coriolis effect.

Northern hemisphere

Southern hemisphere

Free fall with coriolis force



$$m\ddot{\vec r} = m\vec g + \underbrace{2m\dot{\vec r}\times\vec\Omega}_{\text{coriolis force}}$$

small centrifugal force is included in $\vec g$.

$$\ddot{\vec r} = \vec g + 2\dot{\vec r}\times\vec\Omega \quad , \quad \vec g = -g\hat z$$

$$\dot{\vec r} = (\dot x, \dot y, \dot z), \quad \vec\Omega = (0, \Omega\sin\theta, \Omega\cos\theta)$$

so
$$\dot{\vec r}\times\vec\Omega = (\dot y\Omega\cos\theta - \dot z\Omega\sin\theta,\; -\dot x\Omega\cos\theta,\; \dot x\Omega\sin\theta)$$

so
$$\ddot x = 2\Omega(\dot y\cos\theta - \dot z\sin\theta)$$
$$\ddot y = -2\Omega\dot x\cos\theta$$
$$\ddot z = -g + 2\Omega\dot x\sin\theta$$

1st approximation: ignore Ω.

Then $\ddot{\vec r} = (0, 0, -g)$

$\vec r = (0, 0, h - \frac{1}{2}gt^2)$

2^{nd} approximation

Take the previous solution and substitute:

$$\ddot{x} = 2\Omega g t \sin\theta$$
$$\ddot{y} = \phi$$
$$\ddot{z} = -g$$

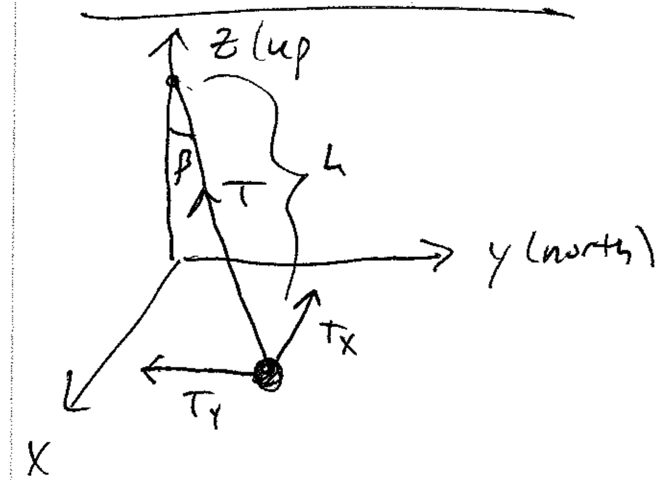
Then $\boxed{x = \frac{1}{3}\Omega g t^3 \sin\theta}$

So the object is deflected in the $(+x)$ direction.
If the object falls 100 m, without drag, at the equator
then $t = \sqrt{2h/g} \approx$ and

$$x = \frac{1}{3}\Omega g\left(\frac{2h}{g}\right)^{3/2}, \quad \Omega = 7.3\times10^{-5}\ \text{sec}^{-1}$$

$$x = \frac{1}{3}(7.3\times10^{-5})(10)(20)^{3/2} \approx 2.2\ \text{cm}$$

Foucalt Pendulum



$$m\ddot{\vec{r}} = \vec{T} + 2m\dot{\vec{r}}\times\vec{\Omega} + m\vec{g}$$

includes centrifugal term

As long as β is small,

$$T_z \approx |\vec{T}| \approx mg \approx T$$

By similar triangles,

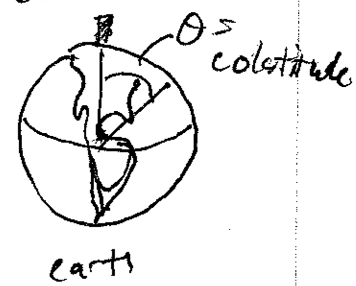
$$\frac{T_x}{T} = -\frac{x}{L} \quad \text{and} \quad \frac{T_y}{T} = -\frac{y}{L}$$

$$\Rightarrow T_x = -\frac{mgx}{L}, \quad T_y = -\frac{mgy}{L}$$

$$\ddot{x} = -\frac{gx}{L} + 2\dot{y}\,\Omega\cos\theta$$

$$\ddot{y} = -\frac{gy}{L} - 2\dot{x}\,\Omega\cos\theta$$

, θ = colatitude



$$\frac{g}{L} = \omega_0^2, \quad \Omega\cos\theta = \Omega_z$$

$$\boxed{\begin{array}{l} \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x = \emptyset \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y = \emptyset \end{array}}$$

<u>Coupled differential equations</u>: Define $\eta = x + iy$

Multiply 2nd Equation by (i) and add:

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = \emptyset$$

Guess solution of the form $\eta(t) = e^{-i\alpha t}$

Then $\alpha^2 - 2\Omega_z\alpha - \omega_0^2 = \emptyset$

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} \approx \Omega_z \pm \omega_0$$

General Solution:

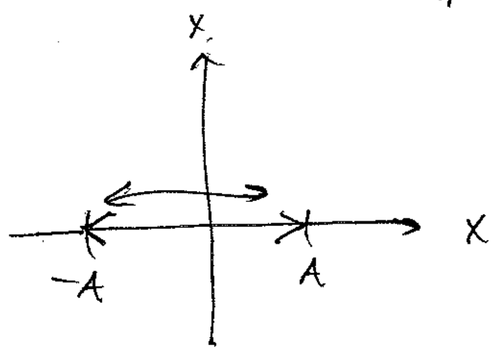
$$\eta = e^{-i\Omega_z t}\left(c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right)$$

Make up some initial conditions:

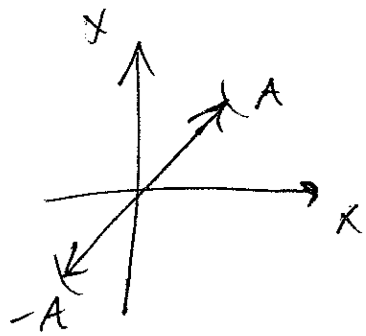
$$x(t=0) = A$$
$$y(t=0) = 0$$
$$v_x(t=0) = 0$$
$$v_y(t=0) = 0$$

Then $\eta(t) = x(t) + i y(t) = A e^{-i\Omega_z t}\cos(\omega_0 t)$

Since Ω_z is small, initially the oscillation is entirely in the x direction:



but after a while



The rate of rotation is $\Omega\cos\theta$, $\theta = 51.0$ for college park, so $\cos\theta \approx 63\%$, so the full period in College Park $\approx$ 1.59 days.