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good

Problem 1

a

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The time-dependent state function is given by the expansion:

$$\psi(\mathbf{r}, t) = \sum_n c_n(t)\, \varphi_n(\mathbf{r})\, e^{-\frac{i E_n t}{\hbar}} \tag{1}$$

where $\varphi_n(\mathbf{r})$ are the time-independent eigenfunctions of the Hamiltonian and $c_n(t)$ are expansion coefficients. For this particular system:

$$\hat{H} = \frac{\Omega}{2}\, \hat{\sigma}_z \tag{2}$$

This Hamiltonian has eigenstates $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, with eigenenergies $E_\uparrow = \frac{\Omega}{2}$ and $E_\downarrow = -\frac{\Omega}{2}$. We approximate the coefficients by:

$$c_n(t) = \sum_k \lambda^k\, c^{(k)}(t) \tag{3}$$

The first order approximation to the coefficient is given by:

$$i\hbar\, \dot{c}_n{}^{(1)}(t) = \sum_k \langle \varphi_n | \hat{H}' | \varphi_k \rangle\, c_k{}^{(0)}(t) \tag{4}$$

If the perturbing Hamiltonian is separable in time and space, then $\langle \varphi_n | \hat{H}' | \varphi_k \rangle = e^{\frac{i(E_n - E_k)t'}{\hbar}} f(t) \langle \varphi_n | \mathbb{H}'(\mathbf{r}) | \varphi_k \rangle$.

$$i\hbar\, \dot{c}_n{}^{(1)}(t) = \sum_k e^{\frac{i(E_n - E_k)t'}{\hbar}} f(t) \langle \varphi_n | \mathbb{H}'(\mathbf{r}) | \varphi_k \rangle\, c_k{}^{(0)}(t) \tag{5}$$

Our system begins in the state $|\uparrow\rangle$. Thus $c_n{}^{(0)}(0) = \delta_{n\uparrow}$:

$$\begin{aligned}
i\hbar\, \dot{c}_n{}^{(1)}(t) &= e^{\frac{i(E_n - E_\uparrow)t'}{\hbar}} f(t) \langle \varphi_n | \mathbb{H}'(\mathbf{r}) | \varphi_\uparrow \rangle \\
\dot{c}_n{}^{(1)}(t) &= \frac{1}{i\hbar}\, e^{\frac{i(E_n - E_\uparrow)t'}{\hbar}} f(t) \langle \varphi_n | \mathbb{H}'(\mathbf{r}) | \varphi_\uparrow \rangle \\
c_n{}^{(1)}(t) &= \frac{1}{i\hbar} \langle \varphi_n | \mathbb{H}'(\mathbf{r}) | \varphi_\uparrow \rangle \int_{-0}^{t} e^{\frac{i(E_n - E_\uparrow)t'}{\hbar}} f(t')\, dt'
\end{aligned} \tag{6}$$

We find the coefficients under the perturbation $f(t) = \alpha\, \theta(t)\, \theta(T - t)$:

$$
\begin{aligned}
c_\downarrow^{(1)}(t) &= \frac{1}{i\hbar}\langle\varphi_\downarrow \| H'(\mathbf{r}) | \varphi_\uparrow\rangle \int_{-0}^{t} e^{\frac{i(E_\downarrow - E_\uparrow)t'}{\hbar}}\,\alpha\,\theta(t')\,\theta(T-t')\,dt' \\
&= \frac{\alpha\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}}{i\hbar}\int_0^t e^{\frac{i(-\frac{\Omega}{2}-\frac{\Omega}{2})t'}{\hbar}}\,dt' \quad ; \quad 0 \le t \le T \\
&= \frac{\alpha}{i\hbar}\left(-\frac{\hbar}{i\Omega}\,e^{-\frac{i\Omega t}{\hbar}}\right)_0^t \\
&= \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right)
\end{aligned}
\tag{7}
$$

The diagonal elements of the perturbation are 0 in our basis so $c_\uparrow^{(1)}(t) = 0$. Then the coefficients are:

$$
\begin{aligned}
c_\uparrow(t) &= c_\uparrow^{(0)}(t) + c_\uparrow^{(1)}(t) = 1 + 0 = 1 \\
c_\downarrow(t) &= c_\downarrow^{(0)}(t) + c_\downarrow^{(1)}(t) = 0 + \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right) = \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right)
\end{aligned}
\tag{8}
$$

and the state function:

$$
\begin{aligned}
\psi(\mathbf{r}, t) &= c_\downarrow(t)\,e^{-\frac{iE_\downarrow t}{\hbar}}\,|\downarrow\rangle + c_\uparrow(t)\,e^{-\frac{iE_\uparrow t}{\hbar}}\,|\uparrow\rangle \\
&= \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right)e^{-\frac{i(-\frac{\Omega}{2})t}{\hbar}}\begin{pmatrix}0\\ 1\end{pmatrix} + e^{-\frac{i(\frac{\Omega}{2})t}{\hbar}}\begin{pmatrix}1\\ 0\end{pmatrix} \\
&= \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{2\hbar}} - e^{\frac{i\Omega t}{2\hbar}}\right)\begin{pmatrix}0\\ 1\end{pmatrix} + e^{-\frac{i\Omega t}{2\hbar}}\begin{pmatrix}1\\ 0\end{pmatrix} \\
&= \begin{pmatrix} e^{-\frac{i\Omega t}{2\hbar}} \\ -\frac{2i\alpha}{\Omega}\sin\left(\frac{\Omega t}{2\hbar}\right)\end{pmatrix}
\end{aligned}
\tag{9}
$$

■ b

The probability the sytem shifts to $|\downarrow\rangle$ is given by:

$$
\begin{aligned}
P_{\uparrow\to\downarrow} &= |c_\downarrow(t)|^2 \\
&= \left(\frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right)\right)\left(\frac{\alpha}{\Omega}\left(e^{\frac{i\Omega t}{\hbar}} - 1\right)\right) \\
&= \frac{\alpha^2}{\Omega^2}\left(1 - e^{-\frac{i\Omega t}{\hbar}} - e^{\frac{i\Omega t}{\hbar}} + 1\right) \\
&= \frac{\alpha^2}{\Omega^2}\left(2 - 2\cos\left(\frac{\Omega t}{\hbar}\right)\right)
\end{aligned}
\tag{10}
$$

■ c

The perturbation can be considered small when $P_{\uparrow\to\downarrow} \ll 1$:

$$
\begin{aligned}
P_{\uparrow\to\downarrow} &\ll 1 \\
\frac{\alpha^2}{\Omega^2}\left(2 - 2\cos\left(\frac{\Omega t}{\hbar}\right)\right) &\ll 1
\end{aligned}
$$

$$\frac{\alpha^2}{\Omega^2}(2-2(-1)) \ll 1$$
$$\frac{4\,\alpha^2}{\Omega^2} \ll 1$$
$$\left|\alpha\right| \ll \frac{\Omega}{2}$$

Problem 2

■ a

The time-dependent Schrödinger equation is given by:

$$\hat{H}\,\psi(\mathbf{r},t) = i\hbar\,\frac{\partial}{\partial t}\,\psi(\mathbf{r},t) \tag{12}$$

Hence:

$$\left(\frac{\Omega}{2}\,\hat{\sigma}_z + \alpha\,\theta(t)\,\theta(T-t)\,\hat{\sigma}_x\right)\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\hbar\,\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$
$$\left(\frac{\Omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\hbar\,\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}\ ;\quad 1 \le t \le T \tag{13}$$
$$\frac{1}{i\hbar}\begin{pmatrix} \frac{\Omega}{2} & \alpha \\ \alpha & -\frac{\Omega}{2} \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

A matrix differential equation of the form $\mathbf{A}\mathbf{x} = \dot{\mathbf{x}}$ has a solution of the form:

$$\mathbf{x} = c_1\,e^{\lambda_1 t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2 t}\,\mathbf{v}_2 \tag{14}$$

where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ its corresponding eigenvector. We determine the eigenvalues and normalized eigenvectors of our system using *Mathematica* :

```
FullSimplify[Eigenvalues[( Ω/2   α
                           α    -Ω/2 )], √ħ² == 0]
```

$$\left\{-\frac{1}{2}\sqrt{4\,\alpha^2+\Omega^2},\ \frac{1}{2}\sqrt{4\,\alpha^2+\Omega^2}\right\}$$

```
FullSimplify[# / Plus @@ #²] & /@ Eigenvectors[( Ω/2   α
                                                 α    -Ω/2 )]
```

$$\left\{\left\{-\frac{\alpha}{\sqrt{4\,\alpha^2+\Omega^2}},\ \frac{1}{2}\left(1+\frac{\Omega}{\sqrt{4\,\alpha^2+\Omega^2}}\right)\right\},\ \left\{\frac{\alpha}{\sqrt{4\,\alpha^2+\Omega^2}},\ \frac{1}{2}-\frac{\Omega}{2\sqrt{4\,\alpha^2+\Omega^2}}\right\}\right\}$$

We define $\omega \equiv \sqrt{4\,\alpha^2+\Omega^2}$, then $\lambda_1 = -\frac{\omega}{2 i\hbar}$, $\lambda_2 = \frac{\omega}{2 i\hbar}$ and $\mathbf{v}_1 = \frac{1}{\omega}\begin{pmatrix} -\alpha \\ \frac{1}{2}(\omega+\Omega) \end{pmatrix}$, $\mathbf{v}_2 = \frac{1}{\omega}\begin{pmatrix} \alpha \\ \frac{1}{2}(\omega-\Omega) \end{pmatrix}$.

$$\begin{aligned}\psi(\mathbf{r},t) &= c_1\, e^{\lambda_1 t}\,\mathbf{v}_1 + c_2\, e^{\lambda_2 t}\,\mathbf{v}_2\\ &= \frac{c_1}{\omega}\, e^{-\frac{\omega t}{2 i \hbar}} \begin{pmatrix} -\alpha \\ \frac{1}{2}(\omega+\Omega)\end{pmatrix} + \frac{c_2}{\omega}\, e^{\frac{\omega t}{2 i \hbar}} \begin{pmatrix} \alpha \\ \frac{1}{2}(\omega-\Omega)\end{pmatrix}\end{aligned} \tag{15}$$

At $t = 0$, the system is in state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This allows us to solve for the constants c_1 and c_2. We use *Mathematica* again:

$$\texttt{Solve}\left[\left\{-\frac{\alpha}{\omega}\,\texttt{c1} + \frac{\alpha}{\omega}\,\texttt{c2} == 1,\ \frac{(\omega+\Omega)}{2\,\omega}\,\texttt{c1} + \frac{(\omega-\Omega)}{2\,\omega}\,\texttt{c2} == 0\right\},\ \{\texttt{c1},\ \texttt{c2}\}\right]$$

$$\left\{\left\{\texttt{c1} \to -\frac{\omega-\Omega}{2\,\alpha},\ \texttt{c2} \to \frac{\omega+\Omega}{2\,\alpha}\right\}\right\}$$

Thus:

$$\psi(\mathbf{r},t) = \frac{1}{2\,\omega}\, e^{\frac{i\omega t}{2\hbar}} \begin{pmatrix} \omega-\Omega \\ -\frac{1}{2\alpha}(\omega^2-\Omega^2)\end{pmatrix} + \frac{1}{2\,\omega}\, e^{-\frac{i\omega t}{2\hbar}} \begin{pmatrix} \omega+\Omega \\ \frac{1}{2\alpha}(\omega^2-\Omega^2)\end{pmatrix} \tag{16}$$

■ b

We use *Mathematica* to Taylor expand the exact solution in α:

$$\psi = \frac{1}{2\,\omega}\, e^{\frac{i\omega t}{2\hbar}} \begin{pmatrix} \omega-\Omega \\ -\frac{1}{2\alpha}(\omega^2-\Omega^2)\end{pmatrix} + \frac{1}{2\,\omega}\, e^{-\frac{i\omega t}{2\hbar}} \begin{pmatrix} \omega+\Omega \\ \frac{1}{2\alpha}(\omega^2-\Omega^2)\end{pmatrix};$$

$$\texttt{FullSimplify}\left[\texttt{Series}\left[\psi\ /.\ \left(\omega \texttt{->} \sqrt{4\,\alpha^2+\Omega^2}\right),\ \{\alpha,\ 0,\ 1\}\right]\right]\ \texttt{// MatrixForm}$$

$$\begin{pmatrix} e^{-\frac{i t \Omega}{2\hbar}} + O[\alpha]^2 \\ -\frac{2\, i\, \texttt{Sin}\left[\frac{t\Omega}{2\hbar}\right]\alpha}{\Omega} + O[\alpha]^2 \end{pmatrix}$$

Thus the first order expansion is:

$$\psi(\mathbf{r},t) \cong \begin{pmatrix} e^{-\frac{i\Omega t}{2\hbar}} \\ -\frac{2 i \alpha}{\Omega}\sin\left(\frac{\Omega t}{2\hbar}\right)\end{pmatrix} \tag{17}$$

This agrees with the solution in Eq. 9.

Problem 3

■ a

We continue from Eq. 6 with the new time-dependent perturbation $f(t) = \alpha\, \frac{t}{T}\, \theta(t)\, \theta(T-t)$

$$c_{\downarrow}(t) = \frac{\langle \downarrow | \hat{\sigma}_x | 1 \rangle}{i\hbar} \int_{-\infty}^{t} e^{\frac{i(E_{\downarrow}-E_1)t'}{\hbar}} \alpha \frac{t'}{T} \theta(t)\theta(T-t)\,dt'$$

$$= \frac{\alpha(0\ \ 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{i\hbar T} \int_0^t t' e^{\frac{i(-\frac{\Omega}{2}-\frac{\Omega}{2})t'}{\hbar}}\,dt' \quad ; \quad 0 \le t \le T$$

$$= \frac{\alpha}{i\hbar T}\left(\left(-\frac{\hbar}{i\Omega}\right)^2\left(-1-\frac{i\Omega}{\hbar}t'\right)e^{-\frac{i\Omega t'}{\hbar}}\Big|_0^t\right)$$

$$= -\frac{\alpha\hbar}{i\Omega^2 T}\left(\left(-1-\frac{i\Omega}{\hbar}t\right)e^{-\frac{i\Omega t}{\hbar}}+1\right)$$

$$= \frac{i\alpha\hbar}{\Omega^2 T}\left(1-e^{-\frac{i\Omega t}{\hbar}}-\frac{i\Omega}{\hbar}t\,e^{-\frac{i\Omega t}{\hbar}}\right)$$

By the same argument used in Problem 1, $c_1(t) = 1\ \forall\ t$. So the perturbed state function to first order is:

$$\psi(\mathbf{r},t) = c_{\downarrow}(t)\,e^{-\frac{iE_{\downarrow}t}{\hbar}}\,|\downarrow\rangle + c_1(t)\,e^{-\frac{iE_1 t}{\hbar}}\,|1\rangle$$

$$= \frac{i\alpha\hbar}{\Omega^2 T}\left(1-e^{-\frac{i\Omega t}{\hbar}}-\frac{i\Omega}{\hbar}t\,e^{-\frac{i\Omega t}{\hbar}}\right)e^{-\frac{i(-\frac{\Omega}{2})t}{\hbar}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\frac{i(\frac{\Omega}{2})t}{\hbar}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{i\alpha\hbar}{\Omega^2 T}\left(e^{\frac{i\Omega t}{2\hbar}}-e^{-\frac{i\Omega t}{2\hbar}}-\frac{i\Omega}{\hbar}t\,e^{-\frac{i\Omega t}{2\hbar}}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\frac{i\Omega t}{2\hbar}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} e^{-\frac{i\Omega t}{2\hbar}} \\ \frac{2\alpha\hbar}{\Omega^2 T}\sin\left(\frac{\Omega t}{2\hbar}\right) + \frac{\alpha t}{\Omega T}e^{-\frac{i\Omega t}{2\hbar}} \end{pmatrix}$$

■ b

The probability the sytem shifts to $|\downarrow\rangle$ is given by:

$$P_{1\to\downarrow} = \left|c_{\downarrow}(t)\right|^2$$

$$= \left(\frac{-i\alpha\hbar}{\Omega^2 T}\left(1-e^{\frac{i\Omega t}{\hbar}}+\frac{i\Omega}{\hbar}t\,e^{\frac{i\Omega t}{\hbar}}\right)\right)\left(\frac{i\alpha\hbar}{\Omega^2 T}\left(1-e^{-\frac{i\Omega t}{\hbar}}-\frac{i\Omega}{\hbar}t\,e^{-\frac{i\Omega t}{\hbar}}\right)\right)$$

$$= \frac{\alpha^2\hbar^2}{\Omega^4 T^2}\left(1-e^{-\frac{i\Omega t}{\hbar}}-\frac{i\Omega}{\hbar}t\,e^{\frac{i\Omega t}{\hbar}}-e^{\frac{i\Omega t}{\hbar}}+1+\frac{i\Omega}{\hbar}t+\frac{i\Omega}{\hbar}t\,e^{\frac{i\Omega t}{\hbar}}-\frac{i\Omega}{\hbar}t+\frac{\Omega^2}{\hbar^2}t^2\right) \tag{20}$$

$$= \frac{\alpha^2\hbar^2}{\Omega^4 T^2}\left(2-e^{\frac{i\Omega t}{\hbar}}-e^{-\frac{i\Omega t}{\hbar}}+\frac{\Omega^2}{\hbar^2}t^2\right)$$

$$= \frac{\alpha^2\hbar^2}{\Omega^4 T^2}\left(2-2\cos\left(\frac{\Omega t}{\hbar}\right)+\frac{\Omega^2}{\hbar^2}t^2\right)$$

■ c

The perturbation can be considered small when $P_{1\to\downarrow} \ll 1$. We work in the regime that $t \ll T$, so that $\frac{t^2}{T^2}$ is negligible:

$$
\begin{aligned}
P_{\uparrow\to\downarrow} &\ll 1 \\
\frac{\alpha^2 \hbar^2}{\Omega^4 T^2}\left(2 - 2\cos\!\left(\frac{\Omega t}{\hbar}\right) + \frac{\Omega^2}{\hbar^2} t^2\right) &\ll 1 \\
\frac{\alpha^2 \hbar^2}{\Omega^4 T^2}(2 - 2(-1)) + \frac{\alpha^2}{\Omega^2}\frac{t^2}{T^2} &\ll 1 \\
\frac{4\alpha^2 \hbar^2}{\Omega^4 T^2} &\ll 1 \\
\left|\alpha\right| &\ll \frac{\Omega^2 T}{2\hbar}
\end{aligned}
$$

Problem 4

■ a

We continue from Eq. 6 with the new time-dependent perturbation $f(t) = \alpha \sin(\omega t)\,\theta(t)\,\theta(T - t)$

$$
\begin{aligned}
c_\downarrow(t) &= \frac{\langle \downarrow | \hat{\sigma}_x | \uparrow \rangle}{i\hbar} \int_{-\infty}^{t} e^{\frac{i(E_\downarrow - E_\uparrow)t'}{\hbar}} \alpha \sin(\omega t')\,\theta(t)\,\theta(T-t)\, dt' \\
&= \frac{\alpha \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{i\hbar} \int_0^t \left(\frac{e^{i\omega t'} - e^{-i\omega t'}}{2i}\right) e^{\frac{i(-\frac{\Omega}{2} - \frac{\Omega}{2})t'}{\hbar}}\, dt' \quad ; \quad 0 \le t \le T \\
&= -\frac{\alpha}{2\hbar} \int_0^t e^{i(\omega - \frac{\Omega}{\hbar})t'} - e^{-i(\omega + \frac{\Omega}{\hbar})t'}\, dt' \quad ; \quad 0 \le t \le T \\
&= -\frac{\alpha}{2\hbar}\left(\frac{1}{i(\omega - \frac{\Omega}{\hbar})} e^{i(\omega - \frac{\Omega}{\hbar})t'} + \frac{1}{i(\omega + \frac{\Omega}{\hbar})} e^{-i(\omega + \frac{\Omega}{\hbar})t'} \Bigg|_0^t\right) \\
&= \frac{\alpha i}{2}\left(\frac{1}{\hbar\omega - \Omega} e^{i(\omega - \frac{\Omega}{\hbar})t} + \frac{1}{\hbar\omega + \Omega} e^{-i(\omega + \frac{\Omega}{\hbar})t} - \frac{1}{\hbar\omega - \Omega} - \frac{1}{\hbar\omega + \Omega}\right) \\
&= \frac{\alpha \hbar \omega i}{\hbar^2 \omega^2 - \Omega^2}\left(\frac{1 + \frac{\Omega}{\hbar\omega}}{2} e^{i(1 - \frac{\Omega}{\hbar})\omega t} + \frac{1 - \frac{\Omega}{\hbar\omega}}{2} e^{-i(1 + \frac{\Omega}{\hbar\omega})\omega t} - 1\right)
\end{aligned}
\tag{22}
$$

We define $\Omega_+ \equiv 1 + \frac{\Omega}{\hbar\omega}$ and $\Omega_- \equiv 1 - \frac{\Omega}{\hbar\omega}$. Then:

$$
c_\downarrow(t) = \frac{\alpha i}{\hbar \omega \Omega_+ \Omega_-}\left(\frac{\Omega_+}{2} e^{i\Omega_- \omega t} + \frac{\Omega_-}{2} e^{-i\Omega_+ \omega t} - 1\right) \tag{23}
$$

By the same argument used in Problem 1, $c_\uparrow(t) = 1 \;\forall\, t$. So the perturbed state function to first order is:

$$
\begin{aligned}
\psi(\mathbf{r}, t) &= c_\downarrow(t)\, e^{-\frac{iE_\downarrow t}{\hbar}} | \downarrow \rangle + c_\uparrow(t)\, e^{-\frac{iE_\uparrow t}{\hbar}} | \uparrow \rangle \\
&= \frac{\alpha i}{\hbar \omega \Omega_+ \Omega_-}\left(\frac{\Omega_+}{2} e^{i\Omega_- \omega t} + \frac{\Omega_-}{2} e^{-i\Omega_+ \omega t} - 1\right) e^{-\frac{i(-\frac{\Omega}{2})t}{\hbar}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\frac{i(\frac{\Omega}{2})t}{\hbar}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$= \frac{\alpha\, i}{\hbar\, \omega\, \Omega_+\, \Omega_-} \left(\frac{\Omega_+}{2}\, e^{i\left(\frac{1}{2}+\frac{1}{2}-\frac{\Omega}{2\hbar\omega}\right)\omega\, t} + \frac{\Omega_-}{2}\, e^{-i\left(\frac{1}{2}+\frac{1}{2}+\frac{\Omega}{2\hbar\omega}\right)\omega\, t} - e^{\frac{i\Omega t}{2\hbar}} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\frac{i\Omega t}{2\hbar}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\alpha\, i}{\hbar\, \omega\, \Omega_+\, \Omega_-} \left(\frac{\Omega_+}{2}\, e^{\frac{i}{2}(1+\Omega_-)\omega\, t} + \frac{\Omega_-}{2}\, e^{-\frac{i}{2}(1+\Omega_+)\omega t} - e^{\frac{i\Omega t}{2\hbar}} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-\frac{i\Omega t}{2\hbar}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{i\Omega t}{2\hbar}} \\ \frac{\alpha i}{\hbar\omega\Omega_+\Omega_-} \left(\frac{\Omega_+}{2}\, e^{\frac{i}{2}(1+\Omega_-)\omega\, t} + \frac{\Omega_-}{2}\, e^{-\frac{i}{2}(1+\Omega_+)\omega t} - e^{\frac{i\Omega t}{2\hbar}} \right) \end{pmatrix}$$

■ b

The probability the sytem shifts to $|\downarrow\rangle$ is given by (keeping in mind that $\Omega_+ + \Omega_- = 2$):

$$\begin{aligned} P_{\uparrow\to\downarrow} = {} & \frac{\alpha\, i}{\hbar\, \omega\, \Omega_+\, \Omega_-} \left(\frac{\Omega_+}{2}\, e^{i\Omega_-\,\omega t} + \frac{\Omega_-}{2}\, e^{-i\Omega_+\,\omega t} - 1 \right) * \\ & \frac{-\alpha\, i}{\hbar\, \omega\, \Omega_+\, \Omega_-} \left(\frac{\Omega_+}{2}\, e^{-i\Omega_-\,\omega t} + \frac{\Omega_-}{2}\, e^{i\Omega_+\,\omega t} - 1 \right) \\ = {} & \frac{\alpha^2}{\hbar^2\, \omega^2\, \Omega_+{}^2\, \Omega_-{}^2} \left(\frac{\Omega_+{}^2}{4} + \frac{\Omega_-{}^2}{4} + 1 + \frac{\Omega_+\,\Omega_-}{4}\, e^{2i\omega t} + \frac{\Omega_+\,\Omega_-}{4}\, e^{-2i\omega t} - \frac{\Omega_+}{2}\, e^{i\Omega_-\,\omega t} \right. \\ & \left. - \frac{\Omega_+}{2}\, e^{-i\Omega_-\,\omega t} - \frac{\Omega_-}{2}\, e^{-i\Omega_+\,\omega t} - \frac{\Omega_-}{2}\, e^{i\Omega_+\,\omega t} \right) \\ = {} & \frac{\alpha^2}{\hbar^2\, \omega^2\, \Omega_+{}^2\, \Omega_-{}^2} \left(\frac{\Omega_+{}^2}{4} + \frac{\Omega_-{}^2}{4} + 1 + \frac{\Omega_+\,\Omega_-}{2} \cos(2\,\omega\, t) - \Omega_+ \cos(\Omega_-\,\omega\, t) - \Omega_- \cos(\Omega_+\,\omega\, t) \right) \end{aligned} \tag{25}$$

■ c

The perturbation can be considered small when $P_{\uparrow\to\downarrow} \ll 1$. All trigonometric functions are estimated at their maxima (1) to maximize the probability:

$$\begin{aligned} P_{\uparrow\to\downarrow} &\ll 1 \\ \frac{\alpha^2}{\hbar^2\, \omega^2\, \Omega_+{}^2\, \Omega_-{}^2} \left(\frac{\Omega_+{}^2}{4} + \frac{\Omega_-{}^2}{4} + 1 + \frac{\Omega_+\,\Omega_-}{2} + \Omega_+ + \Omega_- \right) &\ll 1 \\ \frac{\alpha^2}{\hbar^2\, \omega^2\, \Omega_+{}^2\, \Omega_-{}^2} \left(\left(\frac{\Omega_+}{2} + \frac{\Omega_-}{2} \right)^2 - \frac{\Omega_+\,\Omega_-}{2} + 1 + 1 \right) &\ll 1 \\ \frac{\alpha^2}{\hbar^2\, \omega^2\, \Omega_+{}^2\, \Omega_-{}^2} \left(3 - \frac{\Omega_+\,\Omega_-}{2} \right) &\ll 1 \\ \left| \alpha \right| &\ll \frac{\hbar\, \omega\, \Omega_+\, \Omega_-}{\sqrt{3 - \frac{\Omega_+\,\Omega_-}{2}}} \end{aligned} \tag{26}$$

Problem 5

b

a

The second order appoximation to $c_n(t)$ is given by:

$$i\hbar\,\dot{c}_n^{(2)}(t) = \sum_k \langle\varphi_n\,|\hat{H}'|\,\varphi_k\rangle\, c_n^{(1)}(t) \tag{27}$$

For our particular system, the perturbing Hamiltonian is separable. Hence:

$$i\hbar\,\dot{c}_n^{(2)}(t) = \sum_k \langle\varphi_n\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_k\rangle\, e^{\frac{i(E_n-E_l)t}{\hbar}}\, c_k^{(1)}(t)\, f(t) \tag{28}$$

$$c_n^{(2)}(t) = \frac{1}{i\hbar}\int_{-\infty}^{t}\sum_k \langle\varphi_n\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_k\rangle\, e^{\frac{i(E_n-E_l)t}{\hbar}}\, c_k^{(1)}(t)\, f(t)\, dt \tag{29}$$

We determined earlier that:

$$\begin{gathered} c_\downarrow^{(1)}(t) = \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right) \\ c_\uparrow^{(1)}(t) = 0 \end{gathered} \tag{30}$$

This yields:

$$c_\downarrow^{(2)}(t) = \frac{1}{i\hbar}\int_{-\infty}^{t} \langle\varphi_\downarrow\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_\uparrow\rangle\, e^{\frac{i(E_\downarrow-E_\uparrow)t}{\hbar}}\, c_\uparrow^{(1)}(t)\, f(t) + \langle\varphi_\downarrow\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_\downarrow\rangle\, e^{\frac{i(E_\downarrow-E_\downarrow)t}{\hbar}}\, c_\downarrow^{(1)}(t)\, f(t)\, dt \tag{31}$$

Since diagonal elements of the Hamiltonian are 0 and $c_\uparrow^{(1)}(t) = 0$, the integrand is 0, hence:

$$c_\downarrow^{(2)}(t) = 0 \tag{32}$$

Continuing:

$$\begin{aligned} c_\uparrow^{(2)}(t) &= \frac{1}{i\hbar}\int_{-\infty}^{t} \langle\varphi_\uparrow\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_\uparrow\rangle\, e^{\frac{i(E_\uparrow-E_\uparrow)t}{\hbar}}\, c_\uparrow^{(1)}(t)\, f(t) + \langle\varphi_\uparrow\,|\hat{\mathrm{H}}(\mathbf{r})|\,\varphi_\downarrow\rangle\, e^{\frac{i(E_\uparrow-E_\downarrow)t}{\hbar}}\, c_\downarrow^{(1)}(t)\, f(t)\, dt \\ &= \frac{1}{i\hbar}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\int_{-\infty}^{t} e^{\frac{i\left(\frac{\Omega}{2}-\left(-\frac{\Omega}{2}\right)\right)t'}{\hbar}}\, \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t'}{\hbar}} - 1\right)\alpha\,\theta(t')\,\theta(T-t')\, dt \\ &= \frac{\alpha^2}{i\hbar\Omega}\int_0^t \left(1 - e^{\frac{i\Omega t'}{\hbar}}\right) dt' \;;\;\; 0 \le t \le T \\ &= \frac{\alpha^2}{i\hbar\Omega}\left(t' - \frac{\hbar}{i\Omega}\, e^{\frac{i\Omega t'}{\hbar}}\right)_0^t \\ &= \frac{\alpha^2}{i\hbar\Omega}\left(t - \frac{\hbar}{i\Omega}\, e^{\frac{i\Omega t}{\hbar}} + \frac{\hbar}{i\Omega}\right) \\ &= \frac{\alpha^2}{\Omega^2}\left(e^{\frac{i\Omega t}{\hbar}} - 1\right) - \frac{i\alpha^2}{\hbar\Omega}t \end{aligned} \tag{33}$$

Then:

$$
\begin{aligned}
c_\uparrow(t) &= c_\uparrow^{(0)}(t) + c_\uparrow^{(1)}(t) + c_\uparrow^{(2)}(t) = 1 + 0 = 1 + \frac{\alpha^2}{\Omega^2}\left(e^{\frac{i\Omega t}{\hbar}} - 1\right) - \frac{i\alpha^2}{\hbar\,\Omega}t \\
c_\downarrow(t) &= c_\downarrow^{(0)}(t) + c_\downarrow^{(1)}(t) + c_\downarrow^{(2)}(t) = 0 + \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right) + 0 = \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right)
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\psi(\mathbf{r}, t) &= c_\downarrow(t)\, e^{-\frac{iE_\downarrow t}{\hbar}} \mid \downarrow\rangle + c_\uparrow(t)\, e^{-\frac{iE_\uparrow t}{\hbar}} \mid \uparrow\rangle \\
&= \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{\hbar}} - 1\right) e^{-\frac{i\left(-\frac{\Omega}{2}\right)t}{\hbar}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(1 + \frac{\alpha^2}{\Omega^2}\left(e^{\frac{i\Omega t}{\hbar}} - 1\right) - \frac{i\alpha^2}{\hbar\,\Omega}t\right) e^{-\frac{i\left(\frac{\Omega}{2}\right)t}{\hbar}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{\alpha}{\Omega}\left(e^{-\frac{i\Omega t}{2\hbar}} - e^{\frac{i\Omega t}{2\hbar}}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(\frac{\alpha^2}{\Omega^2}\left(e^{\frac{i\Omega t}{2\hbar}} - e^{-\frac{i\Omega t}{2\hbar}}\right) + \left(1 - \frac{i\alpha^2}{\hbar\,\Omega}t\right) e^{-\frac{i\Omega t}{2\hbar}}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2i\alpha^2}{\Omega^2}\sin\left(\frac{\Omega t}{2\hbar}\right) + \left(1 - \frac{i\alpha^2}{\hbar\,\Omega}t\right) e^{-\frac{i\Omega t}{2\hbar}} \\ -\frac{2i\alpha}{\Omega}\sin\left(\frac{\Omega t}{2\hbar}\right) \end{pmatrix}
\end{aligned} \tag{35}
$$

■ b

Since $c_\downarrow(t)$ is left unchanged by the second-order approximation, $P_{\uparrow\to\downarrow}$ remains unchanged:

$$
P_{\uparrow\to\downarrow} = \frac{\alpha^2}{\Omega^2}\left(2 - 2\cos\left(\frac{\Omega\, t}{\hbar}\right)\right) \tag{36}
$$

■ c

We use *Mathematica* to calculate the second-order Taylor expansion of the exact solution:

```
FullSimplify[Series[ψ /. (ω -> √(4 α² + Ω²)), {α, 0, 2}]] // MatrixForm
```

$$
\begin{pmatrix} e^{-\frac{i t \Omega}{2\hbar}} + \frac{e^{-\frac{i t \Omega}{2\hbar}}\left(-i\, t\, \Omega + \left(-1 + e^{\frac{i t \Omega}{\hbar}}\right)\hbar\right)\alpha^2}{\Omega^2\,\hbar} + O[\alpha]^3 \\ -\frac{2\, i\, \mathrm{Sin}\left[\frac{t\Omega}{2\hbar}\right]\alpha}{\Omega} + O[\alpha]^3 \end{pmatrix}
$$

Simplifying the expression yields:

$$
\begin{aligned}
\psi(\mathbf{r}, t) &= \begin{pmatrix} e^{-\frac{it\Omega}{2\hbar}} + \frac{e^{-\frac{it\Omega}{2\hbar}}\left(-it\Omega + \left(-1 + e^{\frac{it\Omega}{\hbar}}\right)\hbar\right)\alpha^2}{\Omega^2\,\hbar} + O[\alpha]^3 \\ -\frac{2i\,\mathrm{Sin}\left[\frac{t\Omega}{2\hbar}\right]\alpha}{\Omega} + O[\alpha]^3 \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - \frac{i\alpha^2}{\Omega\hbar}\,t\right) e^{-\frac{it\Omega}{2\hbar}} + \frac{\alpha^2}{\Omega^2}\left(-e^{-\frac{it\Omega}{2\hbar}} + e^{\frac{it\Omega}{2\hbar}}\right) + O[\alpha]^3 \\ -\frac{2i\alpha}{\Omega}\sin\left(\frac{\Omega t}{2\hbar}\right) + O[\alpha]^3 \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - \frac{i\alpha^2}{\Omega\hbar}\,t\right) e^{-\frac{it\Omega}{2\hbar}} + \frac{2i\alpha^2}{\Omega^2}\sin\left(\frac{\Omega t}{2\hbar}\right) + O[\alpha]^3 \\ -\frac{2i\alpha}{\Omega}\sin\left(\frac{\Omega t}{2\hbar}\right) + O[\alpha]^3 \end{pmatrix}
\end{aligned} \tag{37}
$$

This is in agreement with our approximation.