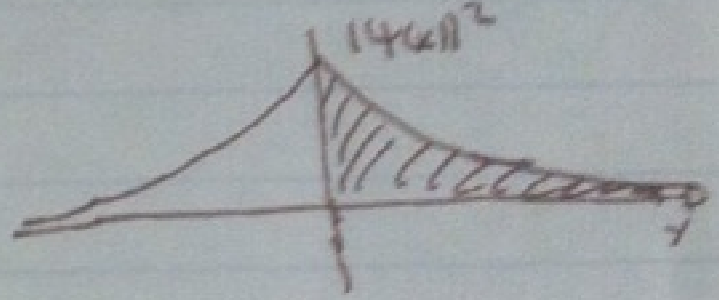
HW 7 - SOLUTIONS
PHY 402 - FALL 2014

I. i) probability $= \int_0^{17} dx \underset{\substack{\uparrow \\ \text{probability density} \\ \text{of finding the particle} \\ @\ x}}{p(x)} = \int_0^{17} dx\, |\underset{\substack{\uparrow \\ \langle x|\psi\rangle}}{\psi(x)}|^2$

$$= \int_0^{17} dx\ 42\, e^{-84|x|} = -\frac{42}{84}\, e^{-84|x|}\Big|_0^{17}$$

$$= -\frac{42}{84}\left[e^{-84\cdot 17} - 1\right] \simeq \frac{1}{2}$$



ii) $$|\psi\rangle = \int_{-\infty}^{\infty} dx\ \sqrt{42}\, e^{-42|x|}\ |x\rangle$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\ \sqrt{2}\, e^{-42|x|}\ |p\rangle \overbrace{\langle p|x\rangle}^{e^{ipx/\hbar}/\sqrt{2\pi\hbar}}$$

$$= \int_{-\infty}^{\infty} dp \underbrace{\left[\int_{-\infty}^{\infty} dx\, \sqrt{42}\, \frac{e^{-42|x| + ipx/\hbar}}{\sqrt{2\pi\hbar}}\right]}_{\equiv\ \tilde{\psi}(p)} |p\rangle \leftarrow \text{eigenbasis of } \hat{p}$$

~~prob. density of measuring momentum p~~

prob. density of measuring momentum p $= |\tilde{\psi}(p)|^2$

II. ~~Find the matrix of $\hat{H}_1$ & then~~

1) $$\hat{H}_0 = \hat{H}_x + \hat{H}_y + \hat{H}_z = \underbrace{\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2}_{\hat{H}_x} + \underbrace{\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2}{2}\hat{y}^2}_{\hat{H}_y} + \underbrace{\frac{\hat{p}_z^2}{2m} + \frac{m\omega^2}{2}\hat{z}^2}_{\hat{H}_z}$$

$$\hat{H}_0\,|n_x n_y n_z\rangle = \underbrace{\hat{H}_x|n_x\rangle}_{\hbar\omega(n_x+\frac{1}{2})|n_x\rangle}\;\underbrace{\hat{H}_y|n_y\rangle}_{\hbar\omega(n_y+\frac{1}{2})|n_y\rangle}\;\underbrace{\hat{H}_z|n_z\rangle}_{\hbar\omega(n_z+\frac{1}{2})|n_z\rangle}$$

$$= \hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right)|n_x n_y n_z\rangle$$

ground state : $|n_x=0,\ n_y=0,\ n_z=0\rangle$, $E = 3\hbar\omega/2$

1st excited states : $|n_x=1,\ n_y=0,\ n_z=0\rangle$, $|n_x=0,\ n_y=1,\ n_z=0\rangle$, $|n_x=0, n_y=0, n_z=1\rangle$,

$E = 5\hbar\omega/2$

2) We need to compute the matrix of $\hat{H}_1$ in the $|100\rangle, |010\rangle, |001\rangle$ basis

$$\langle 100|\,\lambda\,\delta(\hat{x})\,|100\rangle = \int dx\,dy\,dy\ \langle 100|\underbrace{\lambda\,\delta(\hat{x})\,|xyz\rangle}_{\lambda\,\delta(x)\,|xyz\rangle}\underbrace{\langle xyz|100\rangle}_{\varphi_1(x)\varphi_0(y)\varphi_0(z)}$$

$$= \int dx\,dy\,dz\ \lambda\,\delta(x)\ \varphi_1^*(x)\,\varphi_0^*(y)\,\varphi_0^*(z)\ \varphi_1(x)\,\varphi_0(y)\,\varphi_0(z)$$

$$= \int dy\,dz\ \lambda\underbrace{|\varphi_1(0)|^2}_{=0}\ |\varphi_0(y)|^2\,|\varphi_0(z)|^2$$

$$= 0$$

$$\langle 010|\,\lambda\,\delta(\hat{x})\,|010\rangle = \int \cancel{dx}\,dy\,dz\ \cancel{\lambda\delta(x)}\,\lambda\,\underbrace{|\varphi_0(0)|^2}_{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}}\,|\varphi_1(y)|^2\,|\varphi_0(z)|^2$$

$$= \lambda\sqrt{\frac{m\omega}{\pi\hbar}}$$

$$\langle 001|\,\lambda\,\delta(\hat{x})\,|001\rangle = \lambda\sqrt{m\omega/\pi\hbar}$$

Off-diagonal elements:

$$\langle 100 | \lambda \delta(\hat{z}) | 010 \rangle = \lambda \varphi_1^*(0) \varphi_0(0) \int dy\, dz\ \varphi_0^*(y) \varphi_1(y)\ \varphi_0^*(z) \varphi_1(z)$$

$$= 0$$ (orthogonality of The 1D harmonic oscillator energy eigenstates

$$\langle 100 | \lambda \delta(\hat{z}) | 001 \rangle = 0$$

$$\langle 010 | \lambda \delta(\hat{z}) | 001 \rangle = 0$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda \sqrt{m\omega/\pi\hbar} & 0 \\ 0 & 0 & \lambda \sqrt{m\omega/\pi\hbar} \end{pmatrix}$$

The energy of $|100\rangle$ is not shifted at order λ.
The energies of $|010\rangle$ and $|001\rangle$ are shifted by $\lambda\sqrt{\frac{m\omega}{\pi\hbar}}$.

The ground state is not degenerate so the shift in the energy will be given, up to order λ by:

$$\langle 000 | \hat{H}_1 | 000 \rangle = \int dx\, dy\, dz\ \lambda \delta(x)\ |\varphi_0(x)|^2\ |\varphi_0(y)|^2\ |\varphi_0(z)|^2$$

$$= \lambda \sqrt{\frac{m\omega}{\pi\hbar}}$$

This problem would have been more interesting if the perturbation was $\lambda\, \delta(\hat{z})\, \delta(\hat{y})$ . . . :-(

III. $\langle y|\hat{p}|x\rangle = \langle y|\, i\hbar \frac{d}{dx} |x\rangle = i\hbar \frac{d}{dx} \langle y|x\rangle = i\hbar \frac{d}{dx} \delta(x-y)$

$$\langle y|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle y|\hat{p}|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, i\hbar \frac{d}{dx}\delta(x-y)\,\psi(x)$$

$$= -i\hbar \int_{-\infty}^{\infty} dx\, \frac{d}{dy}\,\delta(x-y)\,\psi(x)$$

$$= -i\hbar \frac{d}{dy} \int_{-\infty}^{\infty} dx\, \delta(x-y)\,\psi(x)$$

$$= -i\hbar \frac{d}{dy}\psi(y)$$

$$\langle y|f(\hat{p})|\psi\rangle = \langle y|\left[f(0) + f'(0)\,\hat{p} + \frac{1}{2!} f''(0)\,\hat{p}^2 + \cdots\right]|\psi\rangle$$

$$= f(0)\,\psi(y) - i\hbar f'(0) \frac{d}{dy}\psi(y) + \frac{1}{2!}(-i\hbar)^2 f''(0) \frac{d^2}{dy^2}\psi(y) + \cdots$$

$$= f\left(-i\hbar \frac{d}{dy}\right)\psi(y)$$

IV. see attached

V. i) $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle\phi|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, \phi^*(x)\,\psi(x)$

ii) $\langle\phi|\hat{A}(\hat{x})|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle\phi|\underbrace{\hat{A}(\hat{x})|x\rangle}_{A(x)|x\rangle}\underbrace{\langle x|\psi\rangle}_{\psi(x)}$

$$= \int_{-\infty}^{\infty} dx\, \phi^*(x)\,A(x)\,\psi(x)$$

iii) $\langle\phi|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle\phi|\underbrace{\hat{p}|x\rangle}_{i\hbar\frac{d}{dx}|x\rangle}\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, i\hbar\, \phi^*(x) \frac{d}{dx}\psi(x)$

iv) Typo here. I meant

$$\langle\phi|f(\hat{p})|\psi\rangle = \int_{-\infty}^{\infty} dx\, \langle\phi|\hat{p}|x\rangle\langle x|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx \, i\hbar \langle \phi | \frac{d}{dx} |x\rangle \, \psi(x)$$

$$\left.\begin{aligned} &= i\hbar \int_{-\infty}^{\infty} dx \left(\frac{d}{dx} \phi^{*}(x) \right) \psi(x) \\ &= -i\hbar \int_{-\infty}^{\infty} dx \, \phi^{*}(x) \frac{d}{dx} \psi(x) \end{aligned}\right\} \text{either way}$$

i) $\langle \phi | \psi \rangle = \int d^3r \, \langle \phi | r \rangle \langle r | \psi \rangle$

$$= \int d^3r \, \phi^{\dagger}(r) \, \psi(r)$$

ii) $\langle \phi | A(\hat{r}) | \psi \rangle = \int d^3r \, \langle \phi | A(\hat{r}) | r \rangle \langle r | \psi \rangle$

$$= \int d^3r \, \langle \phi | r \rangle \, A(r) \, \langle r | \psi \rangle$$

$$= \int d^3r \, \phi^{*}(r) \, A(r) \, \psi(r)$$

i) $\langle \phi | \psi \rangle = \int d^3r \sum_{m=+,-} \langle \phi | r, m \rangle \langle r, m | \psi \rangle$

$$= \int d^3r \sum_{m=+,-} \phi_m^{\dagger}(r) \, \psi_m(r)$$

ii) $\langle \phi | A(\hat{r}) | \psi \rangle = \int d^3r \sum_{m=+,-} \langle \phi | \underbrace{A(\hat{r}) | r, m \rangle}_{A(r) \, |r, m\rangle} \underbrace{\langle r, m | \psi \rangle}_{\psi_m(r)}$

$$= \int d^3r \sum_{m=+,-} \phi_m^{\dagger}(r) \, A(r) \, \psi_m(r)$$

iii) $\langle \phi | A(\hat{r}) \hat{S}_z | \psi \rangle = \int d^3r \sum_{m=+,-} \langle \phi | \underbrace{A(\hat{r}) \hat{S}_z | r, m \rangle}_{A(r) \frac{\hbar}{2} m \, |r, m\rangle} \underbrace{\langle r, m | \psi \rangle}_{\psi_m(r)}$

$$= \int d^3r \sum_{m=+,-} \phi_m(r) \, \frac{\hbar}{2} m \, A(r) \, \psi_m(r)$$