# PHY 402 - HOMEWORK 5 SOLUTION
FALL 2014

A. The Born rule says:

① Expand the state of the system at the time of measurement as a linear combination of eigenvects of the observable being measured. In our case the state is $|\psi\rangle$ and the eigenvects of $\hat{p}$ are $|p\rangle$

$$|\psi\rangle = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|\psi\rangle = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx\, \underbrace{\langle p|x\rangle}_{\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}} \underbrace{\langle x|\psi\rangle}_{\psi(x)} |p\rangle$$

$$= \underbrace{\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)}_{=\tilde{\psi}(p)} \quad |p\rangle$$

② the probability (density) of measuring the momentum $p$ $= |\tilde{\psi}(p)|^2 = \left| \int_{-\infty}^{\infty} dx\, e^{ipx/\hbar} \psi(x) \right|^2$

B. $$\hat{S}_z |\text{singlet}\rangle = \left(\hat{S}_z^{(1)} + \hat{S}_z^{(2)}\right)\left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right)$$

$$= \frac{\hbar}{2}\left[\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right]$$

$$= 0$$

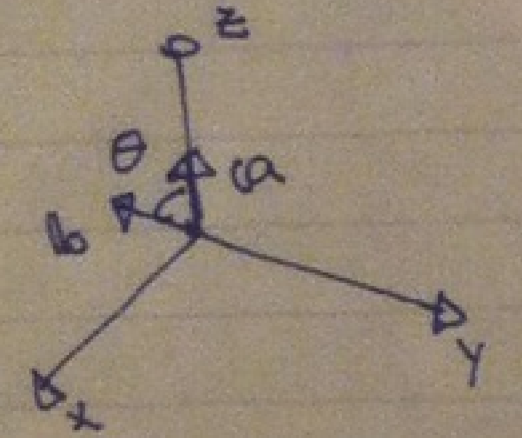
$$\hat{S}^2 |\text{singlet}\rangle = \left(\hat{S}^{(1)2} + \hat{S}^{(2)2} + \hat{S}_+^{(1)} \hat{S}_-^{(2)} + \hat{S}_-^{(1)} \hat{S}_+^{(2)} + 2\hat{S}_z^{(1)} \hat{S}_z^{(2)}\right)\left[\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right]$$

$$= 2\,\frac{3\hbar^2}{4} |\text{singlet}\rangle - \left(\frac{\hbar}{2}\right)^2 \frac{|\uparrow\downarrow\rangle}{\sqrt{2}} + \left(\frac{\hbar}{2}\right)^2 \frac{|\downarrow\uparrow\rangle}{\sqrt{2}}$$
$$+ 2\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$= \left( 2 \times \frac{3\hbar^2}{4} - \frac{\hbar^2}{[illegible]} - 2\,\frac{\hbar^2}{4} \right) |\text{singlet}\rangle$$

$$= 0$$

c. Take the coordinate system such that $\mathbb{a}$ is in the z-direction and $\mathbb{b}$ on the xz-plane:




Then
$$\mathbb{a} = \mathbb{e}_z$$
$$\mathbb{b} = \cos\theta\ \mathbb{e}_z + \sin\theta\ \mathbb{e}_x$$

so $\hat{S}^{(1)} \cdot \mathbb{a} = \hat{S}_z^{(1)}$ and $\hat{S}^{(2)} \cdot \mathbb{b} = \cos\theta\ \hat{S}_z^{(2)} + \sin\theta\ \hat{S}_x^{(2)}$

$$\langle \text{singlet} |\ \hat{S}^{(1)} \cdot \mathbb{a}\ \hat{S}^{(2)} \cdot \mathbb{b}\ | \text{singlet} \rangle = \left[ \frac{\langle \uparrow\downarrow | - \langle \downarrow\uparrow |}{\sqrt{2}} \right] \hat{S}_z^{(1)} \left( \cos\theta\ \hat{S}_z^{(2)} + \sin\theta\ \hat{S}_x^{(2)} \right) \left[ \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right]$$

$$= \frac{1}{2} \left( \langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) \left( \cos\theta\ \hat{S}_z^{(1)} \hat{S}_z^{(2)} + \frac{\sin\theta}{2} \hat{S}_z^{(1)} \left( \hat{S}_+^{(2)} + \hat{S}_-^{(2)} \right) \right) \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

$$= \frac{1}{2} \left( \langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) \left( -\frac{\hbar^2}{4} \cos\theta \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) + \frac{\sin\theta}{2} \frac{\hbar}{2} \hbar\, |\uparrow\uparrow\rangle - \frac{\sin\theta}{2} \left( \frac{\hbar}{2} \right) \hbar\, |\downarrow\downarrow\rangle \right)$$

$$= -\frac{\hbar^2}{8} \cos\theta\ \underbrace{\left( \langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)}_{\underbrace{\langle \uparrow\downarrow | \uparrow\downarrow \rangle}_{=1} - \underbrace{\langle \downarrow\uparrow | \uparrow\downarrow \rangle}_{=0} - \underbrace{\langle \uparrow\downarrow | \downarrow\uparrow \rangle}_{=0} + \underbrace{\langle \downarrow\uparrow | \downarrow\uparrow \rangle}_{=1}}$$

$$= -\frac{\hbar^2}{4} \cos\theta$$

D. i) $V(\hat{x})|\psi\rangle = \left[V(0) + \hat{x}\frac{\partial V}{\partial x}\Big|_{x=0} + \frac{1}{2!}\hat{x}^2\frac{\partial^2 V}{\partial x^2}\Big|_{x=0} + \cdots\right]|\psi\rangle$

$$= \int_{-\infty}^{\infty} dy \left[V(0) + \frac{\partial V}{\partial x}\Big|_{x=0}\hat{x} + \frac{1}{2!}\frac{\partial^2 V}{\partial x^2}\Big|_{x=0}\hat{x}^2 + \cdots\right]|y\rangle\langle y|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dy \underbrace{\left[V(0) + \frac{\partial V}{\partial x}\Big|_{x=0} y + \frac{1}{2!}\frac{\partial^2 V}{\partial x^2}\Big|_{x=0} y^2 + \cdots\right]}_{V(y)}|y\rangle\underbrace{\langle y|\psi\rangle}_{\psi(y)}$$

$$= \int_{-\infty}^{\infty} dy\, V(y)\,\psi(y)\,|y\rangle$$

ii) $\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dy\, \hat{p}|y\rangle\langle y|\psi\rangle = \int_{-\infty}^{\infty} dy\, i\hbar\frac{\partial}{\partial y}|y\rangle\,\psi(y)$

integration by parts $\rightarrow$ $= \int_{-\infty}^{\infty} dy\left(-i\hbar\frac{\partial}{\partial y}\psi(y)\right)|y\rangle$

iii) $\hat{p}^2|\psi\rangle = \hat{p}\,\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} dy\left(-\hbar^2\frac{\partial^2}{\partial y^2}\psi(y)\right)|y\rangle$

$$\left[\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2\right]|\psi\rangle = E|\psi\rangle$$

$$\int_{-\infty}^{\infty} dy\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2}y^2\right]\psi(y)|y\rangle = E\int_{-\infty}^{\infty} dy\underbrace{\langle y|\psi\rangle}_{\psi(y)}|y\rangle$$

or

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{m\omega^2}{2}y^2\right]\psi(y) = E\,\psi(y)$$