PHY 402 - HW4 SOLUTIONS
FALL 2014

A. i) $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

$\mathbf{R} = \dfrac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$ $\Rightarrow$ $\dfrac{m_2 \mathbf{r}}{m_1+m_2} = \dfrac{m_2}{m_1+m_2}(\mathbf{r}_1 - \mathbf{r}_2)$ and $\dfrac{m_1 \mathbf{r}}{m_1+m_2} = \dfrac{m_1}{m_1+m_2}(\mathbf{r}_1 - \mathbf{r}_2)$

$$\Rightarrow \mathbf{R} + \underbrace{\frac{m_2}{m_1+m_2}}_{\frac{\mu}{m_1}}\mathbf{r} = \mathbf{r}_1 \quad \text{and} \quad \mathbf{R} - \underbrace{\frac{m_1}{m_1+m_2}}_{\frac{\mu}{m_2}}\mathbf{r} = \mathbf{r}_2$$

$$\nabla_1 = \frac{\partial}{\partial \mathbf{r}_1} = \underbrace{\frac{\partial \mathbf{R}}{\partial \mathbf{r}_1}}_{\frac{m_1}{m_1+m_2} = \frac{\mu}{m_2}} \frac{\partial}{\partial \mathbf{R}} + \underbrace{\frac{\partial \mathbf{r}}{\partial \mathbf{r}_1}}_{1} \frac{\partial}{\partial \mathbf{r}} = \frac{\mu}{m_2}\nabla_R + \nabla_r$$

$$\nabla_2 = \frac{\partial}{\partial \mathbf{r}_2} = \underbrace{\frac{\partial \mathbf{R}}{\partial \mathbf{r}_2}}_{\frac{m_2}{m_1+m_2} = \frac{\mu}{m_1}} \frac{\partial}{\partial \mathbf{R}} + \underbrace{\frac{\partial \mathbf{r}}{\partial \mathbf{r}_2}}_{-1} \frac{\partial}{\partial \mathbf{r}} = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

ii) $$\Big[\underbrace{-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2}_{} + V(\mathbf{r}_1 - \mathbf{r}_2)\Big]\psi = E\psi$$

$$-\frac{\hbar^2}{2m_1}\left[\frac{\mu}{m_2}\nabla_R + \nabla_r\right]^2 - \frac{\hbar^2}{2m_2}\left[\frac{\mu}{m_1}\nabla_R - \nabla_r\right]^2$$

$$= -\frac{\hbar^2}{2}\left[\frac{\mu^2}{m_1 m_2^2}\nabla_R^2 + \frac{1}{m_1}\nabla_r^2 + \cancel{\frac{2\mu}{m_1 m_2}\nabla_R\nabla_r} + \frac{\mu^2}{m_2 m_1^2}\nabla_R^2 + \frac{1}{m_2}\nabla_r^2 - \cancel{\frac{2\mu}{m_1 m_2}\nabla_R\nabla_r}\right]$$

$$= -\frac{\hbar^2}{2}\underbrace{\frac{\mu^2}{m_1 m_2}\left(\frac{1}{m_2} + \frac{1}{m_1}\right)}_{\frac{1}{m_1+m_2}}\nabla_R^2 - \frac{\hbar^2}{2}\underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{1/\mu}\nabla_r^2$$

$$= -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2$$

iii) $$\left[\frac{-\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right]\varphi_R(R)\,\varphi_r(r) = E\,\varphi_R(R)\,\varphi_r(r)$$

$$\Updownarrow$$

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\,\varphi_R(R) = E_R\,\varphi_R(R)$$

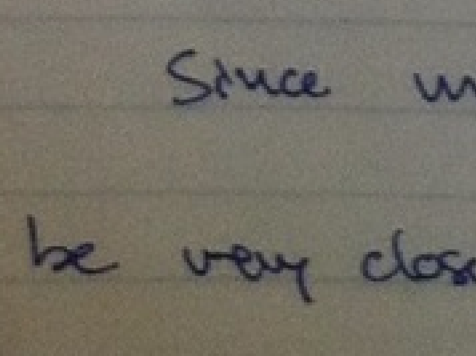
$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right]\varphi_r(r) = E_r\,\varphi_r(r)$$

with $E = E_R + E_r$

iv) In a HCl molecule, with relative separation r between H and Cl, the wave function for the relative motion obeys

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + \underbrace{\frac{K}{2}}_{=\frac{\mu\omega^2}{2}} r^2\right]\varphi(r) = E\,\varphi(r)$$

where we assume an harmonic approximation for the potential energy between H and Cl. and $\mu = \frac{1}{m_H} + \frac{1}{m_{Cl}}$. The photons emitted will have the energy $h\nu$ equal to the difference in energy between two consecutive levels!



$\} \overset{\gamma}{\rightsquigarrow} E = h\nu = \hbar\omega$

so $\nu = \frac{\omega}{2\pi} = \sqrt{\frac{K}{\mu}}\,\frac{1}{2\pi}$.

Since $m_{Cl^{35}}$ is very close to $m_{Cl^{37}}$, $\mu^{35} = \frac{1}{m_H} + \frac{1}{m_{Cl^{35}}}$ will be very close to $\mu^{37} = \frac{1}{m_H} + \frac{1}{m_{Cl^{37}}}$ and there'll be two near photon frequencies split by:

$$\Delta\nu = \frac{1}{2\pi}\sqrt{\frac{K}{\mu^{37}}} - \frac{1}{2\pi}\sqrt{\frac{K}{\mu^{35}}}$$

$$= \frac{\sqrt{K}}{2\pi}\left[\frac{m_H + m_{Cl^{37}}}{m_H m_{Cl^{37}}} - \frac{m_H + m_{Cl^{35}}}{m_H m_{Cl^{35}}}\right]$$

but $m_{Cl^{35}} \simeq 35\, m_H$, $m_{Cl^{37}} \simeq 37\, m_H$ so

$$\Delta\nu \simeq \underbrace{\frac{1}{2\pi}\sqrt{\frac{K}{\mu^{37}}}}_{\nu^{37}}\underbrace{\left[1 - \sqrt{\frac{\mu^{37}}{\mu^{35}}}\right]}$$

$$\simeq 1 - \sqrt{\frac{1+35}{1\times 35}\,\frac{1\times 37}{1+37}}$$

$$\simeq 1 - \sqrt{\frac{36}{35}\,\frac{37}{38}}$$

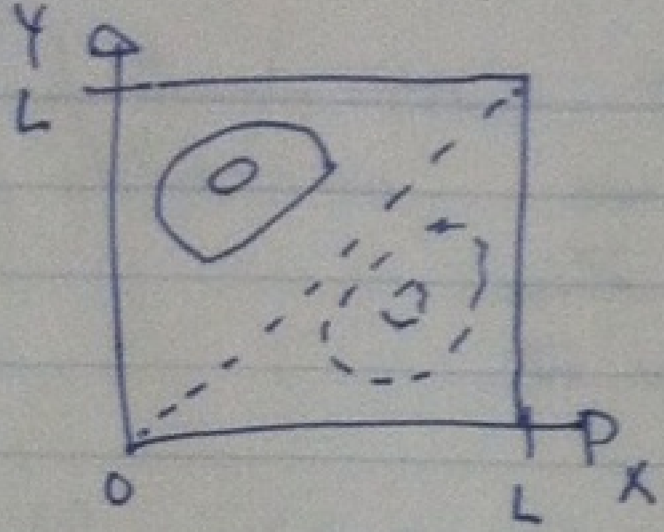
$$\simeq -7.5 \times 10^{-4}$$

1B. one-particle states $\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \ldots$

two-spins up $\Rightarrow$ wave function is symmetric under spin exchange $\Rightarrow$ wave function must be antisymmetric under coordinate exchange

$$\Rightarrow \varphi_{m_1 m_2}(x_1, x_2) = \frac{2}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]\chi^+_{m}\chi^+_{m}$$

The contour plot looks like:



$\psi(x,y)=0$ when $x=y$ (fermions don't like to sit on top of each other)

C. i) $\hat{x}|y\rangle = y|y\rangle \Rightarrow \langle y|\hat{x}|y'\rangle = \langle y|y|y'\rangle = y\langle y|y'\rangle = y\,\delta(y-y')$

ii) $\hat{p}|x\rangle = i\hbar\frac{d}{dx}|x\rangle$ (you can use that as the definition of $\hat{p}$)

$$\Downarrow$$

$$\langle x|\underbrace{\hat{p}}_{=\hat{p}^\dagger} = -i\hbar\frac{d}{dx}\langle x|$$

so $\langle y|\hat{p}|y'\rangle = \langle y|i\hbar\frac{d}{dy'}|y'\rangle = i\hbar\frac{d}{dy}\langle y|y'\rangle = i\hbar\frac{d}{dy'}\delta(y-y')$

$$= -i\hbar\frac{d}{dy}\delta(y-y')$$

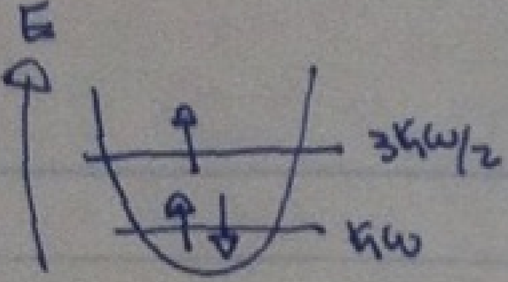
D. i) $\langle p|\hat{x}|p'\rangle = \overbrace{\int_{-\infty}^{\infty}dx}^{=\mathbb{1}}\,\langle p|\hat{x}|x\rangle\langle x|p'\rangle$

$$= \int_{-\infty}^{\infty}dx\; x\,\underbrace{\langle p|x\rangle}_{\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}}\underbrace{\langle x|p'\rangle}_{\frac{e^{-ip'x/\hbar}}{\sqrt{2\pi\hbar}}} = \int_{-\infty}^{\infty}dx\; x\,\frac{e^{i(p-p')x/\hbar}}{(2\pi\hbar)}$$

$$= -i\frac{d}{dp}\int_{-\infty}^{\infty}dx\,\frac{e^{i(p-p')x/\hbar}}{2\pi\hbar}$$

$$= -i\frac{d}{dp}\,\delta(p-p')$$

ii) $\langle p|\hat{p}|p'\rangle = \langle p|p'|p'\rangle = p'\langle p|p'\rangle = p'\,\delta(p-p')$

E. 

The lowest energy configuration would have one spin up electron w/ $n=0$, one spin down electron with $n=0$ and one electron (spin up or down) on the $n=1$ orbital state:

$$\varphi_{m_1 m_2 m_3}(x_1, x_2, x_3) = \varphi_0(x_1)\varphi_0(x_2)\varphi_1(x_3)\chi^+_{m_1}\chi^-_{m_2}\chi^+_{m_3} + \underbrace{\dots}_{\text{terms needed for antisymmetrization}}$$

$$= \begin{vmatrix} \varphi_0(x_1)\chi^+_{m_1} & \varphi_0(x_2)\chi^+_{m_2} & \varphi_0(x_3)\chi^+_{m_3} \\ \varphi_0(x_1)\chi^-_{m_1} & \varphi_0(x_2)\chi^-_{m_2} & \varphi_0(x_3)\chi^-_{m_3} \\ \varphi_1(x_1)\chi^+_{m_1} & \varphi_1(x_2)\chi^+_{m_2} & \varphi_1(x_3)\chi^+_{m_3} \end{vmatrix}$$

$$= \varphi_0(x_1)\varphi_0(x_2)\varphi_1(x_3)\chi^+_{m_1}\chi^-_{m_2}\chi^+_{m_3}$$

$$- \varphi_1(x_1)\varphi_0(x_2)\varphi_0(x_3)\chi^+_{m_1}\chi^-_{m_2}\chi^+_{m_3}$$

$$\vdots$$

$$+ \dots$$

F. Two particles with anti-correlated spins ($|\varphi\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$) are spatially separated and their spin measured.
Every time particle 1 has spin up (down), particle 2 has spin down (up).
Since the result of one measurement could not have been communicated due to the speed of light limit, the two electrons should have definite spin states (and opposite to each other) when they were first separated.
Since quantum mechanics doesn't tell us what those spin states until we measure them, quantum mechanics must be incomplete.