PHY 402 - HW 3 SOLUTION
FALL 2014

A. ii) I'll do ii) first.

eigenvalue

$$\hat{A}|\psi\rangle = a|\psi\rangle \Rightarrow \langle\psi|\hat{A}|\psi\rangle = a\underbrace{\langle\psi|\psi\rangle}_{=1}$$

eigenket

$$a^* = \left(\langle\psi|\hat{A}|\psi\rangle\right)^* = \langle\psi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle = a$$

$\hat{A}$ is hermitian

$$\Downarrow$$

$$\boxed{a \text{ is real}}$$

i)

$$\hat{A}|1\rangle = a_1|1\rangle$$
$$\hat{A}|2\rangle = a_2|2\rangle$$

eigenkets

different eigenvalues
$a_1 \neq a_2$

$$\Downarrow$$

$$\langle 2|\hat{A}|1\rangle = a_1\langle 2|1\rangle$$
$$\langle 1|\hat{A}|2\rangle = a_2\langle 1|2\rangle \Rightarrow \langle 2|\underbrace{\hat{A}^\dagger}_{\hat{A}}|1\rangle = \underbrace{a_2^*}_{=a_2}\langle 2|1\rangle$$

take complex conjugate

$$\Downarrow$$

$$\underbrace{\langle 2|\hat{A}|1\rangle - \langle 2|\hat{A}^\dagger|1\rangle}_{=0} = a_1\langle 2|1\rangle - a_2\langle 2|1\rangle = \underbrace{(a_1-a_2)}_{\neq 0}\langle 2|1\rangle$$

$$\Downarrow$$

$$\langle 2|1\rangle = 0$$

5. i) $\hat{A} = \mathbb{1}\hat{A}\mathbb{1} = \sum_{n,m} |n\rangle\langle n|\hat{A}|m\rangle\langle m| = \sum_{n,m} a_m \underbrace{\langle n|m\rangle}_{\delta_{nm}} |n\rangle\langle m|$

spectral theorem

eigenbasis of $\hat{A}$

$$= \sum_n a_n |n\rangle\langle n|$$

ii) $\hat{A} \sum_n \frac{1}{a_n} |n\rangle\langle n| = \sum_m a_m |m\rangle\langle m| \sum_n \frac{1}{a_n} |n\rangle\langle n|$

$$= \sum_{m,n} a_m \frac{1}{a_n} |m\rangle \underbrace{\langle m|n\rangle}_{\delta_{mn}} \langle n|$$

$$= \sum_n \underbrace{\frac{a_n}{a_n}}_{1} |n\rangle\langle n|$$

$$= \sum_n |n\rangle\langle n|$$

$= \mathbb{1}$ so $\sum_n \frac{1}{a_n}|n\rangle\langle n|$ is indeed the inverse of $\hat{A}$.

$\hat{A}^{-1} = \sum_n \frac{1}{a_n} |n\rangle\langle n|$ would not exist if any of its eigenvalues vanish.

Since $\det\hat{A} = a_1 a_2 \cdots = \prod_n a_n$, $\hat{A}^{-1}$ does not exist if $\det\hat{A} = 0$.

c. i) $[\hat{a}_-, \hat{a}_+] = \left[\frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2\hbar m\omega}}, \frac{m\omega\hat{x} - i\hat{p}}{\sqrt{2\hbar m\omega}} \right]$

$$= \frac{m^2\omega^2}{2\hbar m\omega} \cancel{[\hat{x}, \hat{x}]}^{0} - \frac{i m\omega}{2m\hbar\omega} \overbrace{[\hat{x}, \hat{p}]}^{i\hbar}$$

$$+ \frac{1}{2m\hbar\omega} \cancel{[\hat{p}, \hat{p}]}^{0} + \frac{i m\omega}{2m\hbar\omega} \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar}$$

$$= \frac{2 m\omega\hbar}{2m\hbar\omega} = 1.$$

ii) $\hat{a}_+ + \hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}} 2m\omega\hat{x} \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$

$\hat{a}_+ - \hat{a}_- = \frac{-i}{\sqrt{2m\hbar\omega}} 2 \hat{p} \Rightarrow \hat{p} = \sqrt{\frac{m\hbar\omega}{2}}\, i(\hat{a}_+ - \hat{a}_-)$

$$\langle 0|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle 0|\hat{a}_+ + \hat{a}_-|0\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle 0|\underbrace{\hat{a}_+|1\rangle}_{\sqrt{2}|2\rangle} + \langle 0|\underbrace{\hat{a}_-|0\rangle}_{=0} \right] = 0$$

iii) $\langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega} \langle 0|(\hat{a}_+ + \hat{a}_-)^2|0\rangle$

$$= \frac{\hbar}{2m\omega} \langle 0|\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+|0\rangle$$

$\langle 0|\hat{a}_+^2|0\rangle$
$\langle 0|2\rangle = 0$
$\vdots$

$$\rightarrow = \frac{\hbar}{2m\omega} \underbrace{\langle 0|\hat{a}_-}_{\langle 1|}\underbrace{\hat{a}_+|0\rangle}_{|1\rangle}$$

$$\boxed{= \frac{\hbar}{2m\omega}}$$

iv) $\langle n | \hat{x} | m \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n | \hat{a}_+ + \hat{a}_- | m \rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big[\langle n | \underbrace{\hat{a}_+ | m \rangle}_{\sqrt{m+1}\,|m+1\rangle} + \langle n | \underbrace{\hat{a}_- | m \rangle}_{\sqrt{m}\,|m-1\rangle} \Big]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{m+1}\, \delta_{n,m+1} + \sqrt{m}\, \delta_{n,m-1} \right)$$

D. i) $\langle 1/2 | \hat{S}_z | 1/2 \rangle = \frac{\hbar}{2} \langle 1/2 | 1/2 \rangle = \hbar/2$ $\quad \langle 1/2 | \hat{S}_z | -1/2 \rangle = -\frac{\hbar}{2} \langle 1/2 | -1/2 \rangle = 0$

m

$\langle -1/2 | \hat{S}_z | -1/2 \rangle = -\frac{\hbar}{2} \langle -1/2 | -1/2 \rangle = -\hbar/2$ $\quad \langle -1/2 | \hat{S}_z | 1/2 \rangle = \frac{\hbar}{2} \langle -1/2 | 1/2 \rangle = 0$

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\langle 1/2 | \hat{S}_+ | 1/2 \rangle = 0$

$\langle 1/2 | \hat{S}_+ | -1/2 \rangle = \hbar \underbrace{\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)}}_{1} \langle 1/2 | 1/2 \rangle = \hbar$

$\langle -1/2 | \hat{S}_+ | 1/2 \rangle = 0$

$\langle -1/2 | \hat{S}_+ | -1/2 \rangle = \langle -1/2 | 1/2 \rangle = 0$

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_- = (\hat{S}_+)^\dagger = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\quad \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) = \frac{\hbar}{2i} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

ii) eigenbasis of $\hat{S}_x$: $|L\rangle = \dfrac{|1/2\rangle + |-1/2\rangle}{\sqrt{2}}$ and $|R\rangle = \dfrac{|1/2\rangle - |-1/2\rangle}{\sqrt{2}}$

(worked out in class)

there are ~~[illegible]~~ arbitrary choices made here. We could change the overall phase (for instance, $|L\rangle = \dfrac{-|1/2\rangle - |-1/2\rangle}{\sqrt{2}}$) or order ($|R\rangle = \dfrac{|1/2\rangle + |-1/2\rangle}{\sqrt{2}}$ and $|L\rangle = \dfrac{|1/2\rangle - |-1/2\rangle}{\sqrt{2}}$)

$$\hat{S}_z |L\rangle = \hat{S}_z \left(\frac{|1/2\rangle + |-1/2\rangle}{\sqrt{2}} \right) = \frac{\hbar}{2} \left(\frac{|1/2\rangle - |-1/2\rangle}{\sqrt{2}} \right) = \frac{\hbar}{2} |R\rangle$$

$$\hat{S}_z |R\rangle = \hat{S}_z \left(\frac{|1/2\rangle - |-1/2\rangle}{\sqrt{2}} \right) = \frac{\hbar}{2} \left(\frac{|1/2\rangle + |-1/2\rangle}{\sqrt{2}} \right) = \frac{\hbar}{2} |L\rangle$$

$\langle L|\hat{S}_z|L\rangle = \frac{\hbar}{2}\langle L|R\rangle = 0 \qquad \langle R|\hat{S}_z|L\rangle = \frac{\hbar}{2}\langle R|R\rangle = \frac{\hbar}{2}$

$\langle L|\hat{S}_z|R\rangle = \frac{\hbar}{2}\langle L|L\rangle = 1 \qquad \langle R|\hat{S}_z|R\rangle = \frac{\hbar}{2}\langle R|L\rangle = 0$

$$\hat{S}_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{(in the eigenbasis of } \hat{S}_x\text{!)}$$

$\langle L|\underbrace{\hat{S}_x|L\rangle}_{\hbar/2|L\rangle} = \frac{\hbar}{2}\langle L|L\rangle = \frac{\hbar}{2} \qquad \langle L|\underbrace{\hat{S}_x|R\rangle}_{-\frac{\hbar}{2}|R\rangle} = -\frac{\hbar}{2}\langle L|R\rangle = 0$

$\langle R|\hat{S}_x|L\rangle = 0 \qquad \langle R|\hat{S}_x|R\rangle = -\frac{\hbar}{2}$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(in the eigenbasis of } \hat{S}_x\text{)}$$

I'm too lazy to plow through the calculation of $\hat{S}_y$ so I'll use a dirty trick:

$$i\hbar\hat{S}_y = [\hat{S}_z, \hat{S}_x] \Rightarrow \hat{S}_y = \frac{-i}{\hbar}\left(\hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z\right)$$

$$= \frac{-i}{\hbar}\left[\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= -\frac{i\hbar}{4}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \frac{i\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$