QUANTUM PHYSICS II

PROBLEM SET 3-NEW IMPROVED VERSION

due October 1st, before class

A. Hermitian operators

i) Show that eigenkets of a hermitian operators corresponding to different eigenvalues are orthogonal.

ii) Show that the eigenvalues of a hermitian operators are real.

I'm not going to ask you to prove but it is also true that the set of eigenvectors form a complete set. So, by choosing properly normalized eigenkets, one can form an orthonormal basis, called an eigenbasis of the given hermitian operator. This result is sometimes called the "Spectral Theorem".

B. Bra-ket-ology

i) Use the Spectral Theorem to argue that any hermitian operator $\hat{A}$ can be written as

$$\hat{A} = \sum_n a_n |n\rangle\langle n|, \tag{1}$$

where $|n\rangle$ are its eigenvectors, a_n the corresponding eigenvalues and the sum is over all eigenvectors.

iii) Let $\hat{A} = \sum_n a_n |n\rangle\langle n|$ be an hermitian operator. Show that its inverse is given by $\hat{A}^{-1} = \sum_n \frac{1}{a_n}|n\rangle\langle n|$. What is the condition on $\hat{A}$ so the eigenvalues for the inverse to exist? Hint: what si the determinant of $\hat{A}$?

C. Harmonic oscillator

I hope you learned how to solve the harmonic oscillator using operator methods. If not, you can look it up on Griffiths. The outcome of that discussion is that the eigenvalues/eigenvectors of

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{2}$$

are given by

$$\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle, \qquad |n\rangle = \frac{1}{\sqrt{n!}}\hat{a}_+^n|0\rangle, \tag{3}$$

where

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} \mp i\hat{p}), \tag{4}$$

(the states $|n\rangle$ are already normalized) and the ground state $|0\rangle$ satisfies

$$\hat{a}_-|0\rangle = 0. \tag{5}$$

(The "0" on the right is actually the null ket. I don't want to write it as $|0\rangle$ so it won't be confused with the ground state which is definitely NOT the null ket). Notice that $(\hat{a}_-)^\dagger = \hat{a}_+$.

i) Show that

$$[\hat{a}_-, \hat{a}_+] = 1, \tag{6}$$

using the commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

ii) Compute $\langle 0|\hat{x}|0\rangle$.

iii) Compute $\langle 0|\hat{x}^2|0\rangle$.

iv) Compute the matrix element $\langle n|\hat{x}|m\rangle$.

I. SPIN 1/2 WITH BRAS & KETS

i) Find the matrix representing the operators $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$ in the basis of eigenstates of $\hat{S}_z$ starting from the relations:

$$\hat{S}^2|sm\rangle = \hbar^2 s(s+1)\ |sm\rangle, \tag{7}$$

$$\hat{S}_z|sm\rangle = m\hbar\ |sm\rangle, \tag{8}$$

$$\hat{S}_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\ |sm\pm 1\rangle. \tag{9}$$

ii)Find now the matrix representing the operators $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$ in the basis of eigenstates of $\hat{S}_x$.