PHYS 402 - Fall 2014

HW 2 SOLUTION

A. after the rotation $|\psi\rangle \to \hat{R}(z,\pi)|\psi\rangle = e^{-\frac{i\pi}{\hbar}\hat{S}_z}|\psi\rangle$

$$= e^{-\frac{i\pi}{2}\hat{\sigma}_z}|\psi\rangle$$

$$= (\underbrace{\cos\pi/2}_{=0} - i\hat{\sigma}_z \underbrace{\sin\pi/2}_{=1})|\psi\rangle$$

in matrix notation $-i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{5}\\ 2i/\sqrt{5}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-i\\ -2\end{pmatrix}$

measurement of $\hat{S}_x$: expand $\frac{1}{\sqrt{5}}\begin{pmatrix}-i\\ -2\end{pmatrix}$ in the eigenbasis of $\hat{S}_x = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\right\}$

$$\frac{1}{\sqrt{5}}\begin{pmatrix}-i\\ -2\end{pmatrix} = \alpha\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} + \beta\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\Rightarrow \underbrace{\frac{1}{\sqrt{2}}(1\ 1)\frac{1}{\sqrt{5}}\begin{pmatrix}-i\\ -2\end{pmatrix}}_{-\frac{1}{\sqrt{10}}(2+i)} = \alpha\underbrace{\frac{1}{\sqrt{2}}(1\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}_{=1} + \beta\underbrace{\frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}1\\ -1\end{pmatrix}}_{=0}$$

$$\underbrace{\frac{1}{\sqrt{2}}(1\ -1)\frac{1}{\sqrt{5}}\begin{pmatrix}-i\\ -2\end{pmatrix}}_{\frac{1}{\sqrt{10}}(2-i)} = \alpha\underbrace{\frac{1}{\sqrt{2}}(1\ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}_{=0} + \beta\underbrace{\frac{1}{\sqrt{2}}(1\ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}_{=1}$$

$$\Downarrow$$

$$\alpha = -\frac{2+i}{\sqrt{10}}, \quad \beta = \frac{2-i}{\sqrt{10}}$$

$$\boxed{\text{prob. of } S_x = +\hbar/2 = |\alpha|^2 = \frac{4+1}{10} = \frac{1}{2}, \quad \text{prob. of } S_x = -\hbar/2 = |\beta|^2 = \frac{4+1}{10} = \frac{1}{2}}$$

B. Let us start with $\begin{pmatrix}1\\0\end{pmatrix}$, the eigenspinor of $\hat{S}_z$ w/ eigenvalue $+\frac{\hbar}{2}$, and apply a rotation around y by θ and around z by φ. Presumably, this would create a spinor pointing in the (θ,φ) direction, that is, an eigenspinor of $n\cdot\hat{S}$.

$$\underbrace{\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix}}_{e^{-i\varphi\hat{S}_z/\hbar}} \underbrace{\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}}_{e^{-\frac{i\theta\hat{S}_y}{\hbar}}} \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2\end{pmatrix}$$

Let us now check whether $\begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2\end{pmatrix}$ is an eigenstate of $n\cdot\hat{S}$:

$$n\cdot\hat{S} = n_x\hat{S}_x + n_y\hat{S}_y + n_z\hat{S}_z = \sin\theta\cos\varphi\,\hat{S}_x + \sin\theta\sin\varphi\,\hat{S}_y + \cos\theta\,\hat{S}_z$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}$$

$$\underbrace{\frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}}_{n\cdot\hat{S}} \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2\end{pmatrix} = \begin{pmatrix}\cos\theta\cos\frac{\theta}{2}\, e^{-i\varphi/2} + \sin\theta\sin\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\theta\cos\frac{\theta}{2}\, e^{i\varphi/2} - \cos\theta\sin\frac{\theta}{2}\, e^{i\varphi/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 \\ e^{i\varphi/2}\sin\theta/2\end{pmatrix}$$

eigenvalue

eigenvector

An alternative method would be to find the eigenvectors of $\frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}$ by the usual methods of linear algebra.

C. a) $\hat{H} = \overset{g \simeq 2}{\frac{eg}{2m}} \hat{S} \cdot \mathbb{B} \simeq \frac{e\hbar}{2m} \hat{\sigma}_z B_0 \cos\omega t \simeq \underset{\text{matrix notation}}{\frac{eB_0\hbar}{2m} \cos\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

b) $i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$

In matrix notation and with $\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$

$$i\hbar \begin{pmatrix} \frac{da}{dt} \\ \frac{db}{dt} \end{pmatrix} = \frac{eB_0}{2m} \cos\omega t \begin{pmatrix} a(t) \\ -b(t) \end{pmatrix}$$

or

$$\left.\begin{aligned} i\hbar \frac{d}{dt} a(t) &= \frac{eB_0\hbar}{2m} \cos\omega t \; a(t) \\ i\hbar \frac{d}{dt} b(t) &= -\frac{eB_0\hbar}{2m} \cos\omega t \; b(t) \end{aligned}\right\} \text{uncoupled ODE's}$$

Initial condition: $\psi(t=0) = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{eigenstate of } \hat{S}_x \text{ with eigenvalue } \hbar/2} \Rightarrow a(0) = b(0) = \frac{1}{\sqrt{2}}$

$$\left.\begin{aligned} &a(0) = 1/\sqrt{2} \\ &\frac{d}{dt} a(t) = -\frac{ieB_0}{2m} \cos\omega t \; a(t) \end{aligned}\right\} a(t) = a(0) \, e^{-\frac{ieB_0}{2m} \int_0^t dt' \cos\omega t'} = \frac{1}{\sqrt{2}} e^{-\frac{ieB_0}{2m\omega} \sin\omega t}$$

$$\left.\begin{aligned} &b(0) = 1/\sqrt{2} \\ &\frac{d}{dt} b(t) = \frac{ieB_0}{2m} \cos\omega t \; b(t) \end{aligned}\right\} b(t) = b(0) \, e^{\frac{ieB_0}{2m} \int_0^t dt' \cos\omega t'} = \frac{1}{\sqrt{2}} e^{\frac{ieB_0}{2m\omega} \sin\omega t}$$

So $\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{ieB_0}{2m}\sin\omega t} \\ e^{\frac{ieB_0}{2m}\sin\omega t} \end{pmatrix}$

c) We now expand $\psi(t)$ in the eigenbasis of $\hat{S}_x$, namely, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{eB_0}{2m}\sin\omega t} \\ e^{\frac{ieB_0}{2m}\sin\omega t} \end{pmatrix} = \overbrace{\cos\left(\frac{eB_0}{2m}\sin\omega t\right)}^{\frac{1}{\sqrt{2}}(1\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\cdots}\\ e^{i\cdots}\end{pmatrix}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$+\underbrace{i\sin\left(\frac{eB_0}{2m}\sin\omega t\right)}_{\frac{1}{\sqrt{2}}(1\ -1)\begin{pmatrix}e^{i\cdots}\\ e^{i\cdots}\end{pmatrix}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

prob. of $S_x = +\hbar/2$ $= \cos^2\left(\frac{eB_0}{2m}\sin\omega t\right)$, $\boxed{\text{prob. of } S_x = -\hbar/2 = \sin^2\left(\frac{eB_0}{2m}\sin\omega t\right)}$

d) The probability of $S_x = -\hbar/2$ reaches a maximum when $\frac{eB_0}{2m}\sin\omega t = \frac{\pi}{2}$. The maximum $\sin\omega t$ can be is $\sin\omega t = 1$ (what happens every time $t = \frac{\pi}{\omega}(n+\frac{1}{2})$, $n = 0, 1, 2, \ldots$). At those instants, the probability is $\sin^2\left(\frac{eB_0}{2m}\right)$. For this to be 1 (total reversal of spin), $\frac{eB_0}{2m} \geq \frac{\pi}{2}$ or

$$\boxed{B_0 \geq \pi\frac{m}{e}}$$