QUANTUM PHYSICS I

PROBLEM SET 1

due September 16, before class

PHYS-402

A. Exercise your math muscles

1) compute i^i
2) compute $e^{i\pi/2}$
3) find the general solution to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x), \tag{1}$$

for $E > 0$.

4) what is the solution to the problem above satisfying the conditions

$$\psi(0) = 1, \qquad \frac{d\psi(x)}{dx}|_{x=0} = 0 \ ? \tag{2}$$

5) Solve

$$\frac{df(t)}{dt} = Af(t),\ f(0) = B. \tag{3}$$

6) Solve

$$\frac{d^2y(x)}{dx^2} = Ay(x),\ y(0) = B,\ \frac{dy}{dx}|_{x=0} = 0, \quad A > 0. \tag{4}$$

6) Solve

$$\frac{d^2y(x)}{dx^2} = Ay(x),\ y(0) = B,\ \frac{dy}{dx}|_{x=0} = 0, \quad A < 0. \tag{5}$$

B. All you wanted to know about Dirac's δ-function and were afraid to ask

We can define the δ-function buy its behavior inside integrals:

$$\int_a^b f(x)\delta(x-y) \equiv f(y), \quad \text{for} \quad a < y < b, \tag{6}$$

for any well-behaved function $f(x)$, usually called the *test-function.* You can assume these test-functions are always well behaved and go to zero at infinity. Show that

a) $\int_{-\infty}^{\infty} dx(x^3-1)\delta(x-1) = 0$

b) $\delta(cx) = \frac{1}{|c|}\delta(x)$ (Hint: Insert both sides of the equation in the definition of δ above and change variables.)

c) $\frac{d\theta(x)}{dx} = \delta(x)$ where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x < 0. \end{cases} \tag{7}$$

(and $\theta(0) = 1/2$ if it ever matters).

d) What is the Fourier transform of $\delta(x)$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\delta(x) =? \tag{8}$$

Use Plancherel's theorem (see text) to show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}, \tag{9}$$

which is a relation we used in class after a hand waving "proof".

e) Show that

$$\int_{-\infty}^{\infty} dx f(x)\delta'(x) = -f'(0). \tag{10}$$

Feel free to assume that $f(x) \to$ as fast as necessary as $x \to \pm\infty$.

f) Another way of defining the δ-function is through the relation

$$\delta(x) \equiv \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}. \tag{11}$$

Show that the result of d) is the same using this new definition. Feel free to exchange the order of limits and integrations and assume that the test functions are as well behaved as necessary, we are all friends here.

C. Eigenfunctions/eigenvalues of common operators

a) State what the eigenfunctions and corresponding eigenvalues of the position operator $\hat{x}$ are. Then show that they form an orthonormal and complete set (if the spectrum="set of eigenvalues" is continuous, the normalization condition is $\int dx \phi_y^*(x)\psi_{y'}(x) = \delta(y-y')$).

b) State what the eigenfunctions and corresponding eigenvalues of the momentum operator $\hat{p}$ are. Then show that they form an orthonormal and complete set (if the spectrum="set of eigenvalues" is continuous, the normalization condition is $\int dx \phi_y^*(x)\psi_{y'}(x) = \delta(y-y')$). The proof of the completeness relation here relies on item B.d) above.

c) Wait one day and then repeat items a) and b). Yes, I'm serious. For no extra credit, repeat this every day of the week, just after brushing in the morning.

D. Operator wizardry

The momentum operator is represented, in the position eigenbasis, as $\hat{p} = -i\hbar d/dx$.

a) Compute

$$e^{-i\frac{y}{\hbar}\hat{p}} f(x). \tag{12}$$

Hint: The exponential of an operator is defined, in analogy to the exponential of a number, by $e^{\hat{A}} = \sum_{n=0}^{\infty} \hat{A}^n/n!$. Keep in mind the Taylor expansion of $f(x)$.

and remember the Taylor series expression.

b) Show that $\Psi(x,t) = e^{-i\hat{H}t/\hbar}\Psi(x,0)$ satisfies the time-dependent Schrödinger equation (assuming $\hat{H}$ is time independent). It is said that $\hat{p}$ generates space translations (by item a) and $\hat{H}$ generates time translations (by item b).

by Chunxiao Liu

A.

1) $i^i = e^{i\frac{\pi}{2}\cdot i} = e^{-\frac{\pi}{2}}$

2) $e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$

3) $$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

$\therefore \psi = Ae^{ikx} + Be^{-ikx}$

where $k = \sqrt{2mE/\hbar^2}$

4) $\psi(0) = A + B = 1$

$\left.\frac{d}{dx}\psi\right|_{x=0} = ik(A-B) = 0$ $\qquad \therefore A = B = \frac{1}{2}$

$\therefore \psi = \frac{1}{2}\cdot(e^{ikx} + e^{-ikx}) = \cos kx$

5) $$\frac{df}{f} = A\,dt'$$

$\int_0^t :\quad \ln f\Big|_0^t = At \Rightarrow \frac{f(t)}{f(0)} = e^{At}$

$f(t) = Be^{At}$

~~$\therefore f(t) - f(0) = e^{At}$~~

~~$f(t) = B + e^{At}$~~

6) since $A > 0$

$$y(x) = C e^{\sqrt{A}x} + D e^{-\sqrt{A}x}$$

$$= C' \frac{e^{\sqrt{A}x} + e^{-\sqrt{A}x}}{2} + D' \frac{e^{\sqrt{A}x} - e^{-\sqrt{A}x}}{2}$$

at $x=0$: $y(0) = C' = B$

$$\left.\frac{dy}{dx}\right|_{x=0} = D'\sqrt{A} = 0 \qquad D' = 0$$

$$\therefore y(x) = B \cdot \cosh(\sqrt{A}x)$$

for $A < 0$:

$$y(x) = E\cos(\sqrt{-A}x) + F\sin(\sqrt{-A}x)$$

$$y(0) = E = B$$

$$y'(0) = F\sqrt{-A} = 0 \qquad \therefore F = 0$$

$$\therefore y(x) = B\cos(\sqrt{-A}x)$$

B.

a) $$\int_{-\infty}^{+\infty} dx\, \delta(x-1)\,(x^3-1) = x^3-1\Big|_{x=1} = 0$$

b) 1° if $c>0$

$$I = \int_{-\infty}^{+\infty} dx\, f(x)\,\delta(cx) = \frac{1}{c}\int_{-\infty}^{+\infty} d(cx)\, f\left(\frac{cx}{c}\right)\delta(cx) = \frac{1}{c}\int_{-\infty}^{+\infty} dx'\, f\left(\frac{x'}{c}\right)\delta(x')$$

$$= \frac{1}{c} f(0)$$

2° if $c<0$

$$I = \frac{1}{c}\int_{-\infty}^{+\infty} d(cx)\, f\left(\frac{cx}{c}\right)\delta(cx) = \frac{1}{c}\int_{+\infty}^{-\infty} dx'\, f\left(\frac{x'}{c}\right)\delta(x') = \frac{1}{(-c)}\int_{-\infty}^{+\infty} dx'\, f\left(\frac{x'}{c}\right)\delta(x')$$

$$= \frac{1}{-c} f(0)$$

$$\therefore \int_{-\infty}^{+\infty} dx\, f(x)\,\delta(cx) = \frac{1}{|c|} f(0) = \frac{1}{|c|}\int_{-\infty}^{+\infty} dx\, f(x)\,\delta(x)$$

$$\therefore \delta(cx) = \frac{1}{|c|}\delta(x)$$

c) for $\frac{d\theta(x)}{dx}$, it is zero when $x>0$ or $x<0$, so we only care about its behaviour right near $x=0$

$$\int_{-\infty}^{+\infty} f(x)\frac{d\theta}{dx}x = \cancel{\int_{-\infty}^{+\infty}} \int_{-\varepsilon}^{\varepsilon} f(x)\frac{d\theta}{dx}dx \xlongequal[\text{continous}]{f \text{ is}} f(0)\int_{-\varepsilon}^{\varepsilon}\frac{d\theta}{dx}dx$$

$$= f(0)\,\theta\Big|_{-\varepsilon}^{\varepsilon} = f(0)\cdot 1 = f(0) \qquad \therefore \frac{d\theta}{dx} = \delta(x)$$

d) $F(k) = \int_{-\infty}^{+\infty} dx \left(\frac{e^{-ikx}}{\sqrt{2\pi}} \right) \delta(x) = \frac{1}{\sqrt{2\pi}}$

w/ inverse FFT

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{ikx} \cdot \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{ikx}$$

e) $$\int_{-\infty}^{+\infty} dx\, f(x) \frac{d\delta(x)}{dx} = \cancel{f(x)\delta(x)\Big|_{-\infty}^{+\infty}}^{\,0} - \int_{-\infty}^{+\infty} dx \frac{df}{dx} \delta(x) = -f'(0)$$

f) $$\cancel{\int_{-\infty}^{+\infty} dx\, f(x) \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} (-2\alpha x)}$$

$$I = \int_{-\infty}^{+\infty} dx\, f(x) \frac{d}{dx}\left(\lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \right)$$

$$= \underbrace{f(x) \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \Big|_{-\infty}^{+\infty}}_{\to 0} - \lim_{\alpha\to\infty} \int_{-\infty}^{+\infty} dx\, f'(x) \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}$$

since $\alpha\to\infty$, $e^{-\alpha x^2}$ decays very quickly from $x=0$.

$$I \simeq -\lim_{\alpha\to\infty} f'(0) \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} = -f'(0) \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} \cdot \sqrt{\frac{\pi}{\alpha}} = -f'(0)$$

C a) eig. fn : $\delta(x-x')$

eig. value : x'

ortho : $\langle x''|x'\rangle = \int_{-\infty}^{+\infty} dx\, \delta(x-x'')\,\delta(x-x') = \delta(x'-x'')$

~~complete: $\int dx' |x'\rangle\langle x'|$~~ =

$\forall f(x)$: $f(x) = \int_{-\infty}^{+\infty} f(x')\,\delta(x'-x)\,dx'$

$\therefore \{\delta(x'-x)\}$ is complete

b) eig. fn $\frac{1}{\sqrt{2\pi}} e^{ikx}$

eig. val : $\hbar k$

$$\langle k|k'\rangle = \int \frac{dx}{2\pi} e^{i(k'-k)x} = \delta(k'-k)$$

$\forall f(x)$:

$$f(x) = \int dx' f(x')\,\delta(x'-x) = \underline{\int_{-\infty}^{+\infty} dx' f(x')} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(\underline{x'}-x)}$$

$$= \int \tilde{F}(k) \frac{1}{\sqrt{2\pi}} e^{-ikx}\, dk \qquad \therefore \{e^{-ikx}\} \text{ is complete}$$

D.

a) $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$

$$\therefore e^{-i\frac{y}{\hbar}\frac{\hbar}{i}\frac{d}{dx}} f(x) = e^{-y\frac{d}{dx}} f(x)$$

$$= \sum_{n=0}^{\infty} \left(-y\frac{d}{dx}\right)^n f(x) = \sum_{n=0}^{\infty} \frac{d^n f}{dx^n}(-y)^n = f(x-y)$$

b) t-dep.t S.E.

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\,\Psi(x,t)$$

$$i\hbar\frac{\partial}{\partial t} e^{-iHt/\hbar}\,\Psi(x,0) = i\hbar\left(-i\frac{H}{\hbar}\right)e^{-iHt/\hbar}\,\Psi(x,0)$$

$$= H\,\Psi(x,t)$$