ACCELERATED ADDITION OF ANGULAR MOMENTA (INCLUDING SPIN)

Simplest case: two particles with spin = 1/2. We saw that a generic spin-1/2 state can be described as a linear superposition of two states, the spin-up and spin-down states, which are a very convenient choice because they are eigenstates of both S^2 and S_z.

How many possible states can my 2-particle system assume? The first particle can be either $\uparrow$ or $\downarrow$, the same goes for the second particle. I have a total of 4 possible states, corresponding to the possible combinations of quantum numbers m_1 and m_2, the eigenvalues of $S_z^{(1)}$ (S_z applied to the first particle) and $S_z^{(2)}$, respectively.

χ_1 = state of particle-1, eigenstate of $S_z^{(1)}$ with eigenvalue $m_1\hbar$

$$S_z^{(1)}\chi_1 = \hbar m_1 \chi_1$$

χ_2 = same as above, for particle-2

Then, m_1 could be $\pm 1/2$, $m_2 = \pm 1/2$; I can describe any 2-particle state using a linear combination of four 2-particle states:

$\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$.

The question I want to ask myself is: I see I can use m_1 and m_2 to define 2-particle states that are eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$, and use those <u>four</u> states ($\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$) as a basis to represent any generic 2-particle states
spin-1/2 particles, of course

What about the value of the total angular momentum of these 2-particle states? Can I find a probably <u>different</u> set of states that instead of being eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$ are eigenstates of J^2 and J_z, where $J^2 = (\vec{S}^{(1)} + \vec{S}^{(2)})^2$ and $J_z = S_z^{(1)} + S_z^{(2)}$?

The answer is yes, and it is not hard to find these new 2-particle states.

First of all, let's look at J_z, and how it works on our four 2-particle states that are also eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$:

$$\chi_1 = \uparrow \text{ or } \downarrow \quad \text{and} \quad S_z^{(1)} \chi_1 = \hbar m_1 \chi_1 \qquad (m_1 = \pm 1/2)$$
$$\chi_2 = \uparrow \text{ or } \downarrow \quad \text{and} \quad S_z^{(2)} \chi_2 = \hbar m_2 \chi_2 \qquad (m_2 = \pm 1/2)$$

Then:

$$J_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2 = S_z^{(1)} \chi_1 \chi_2 + \chi_1 S_z^{(2)} \chi_2 =$$

$S_z^{(2)}$ does nothing to particle-1, it operates only on particle-2

$$= \hbar m_1 \chi_1 \chi_2 + \hbar m_2 \chi_1 \chi_2$$
$$= \hbar (m_1 + m_2) \chi_1 \chi_2$$

This tells us that our four states $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ are <u>already</u> also eigenstates of J_z, with eigenvalues:

$$J_z |\uparrow\uparrow\rangle = \hbar \left(\frac{1}{2} + \frac{1}{2}\right) |\uparrow\uparrow\rangle = \hbar |\uparrow\uparrow\rangle \qquad m_j = 1$$
$$J_z |\uparrow\downarrow\rangle = \hbar \left(\frac{1}{2} - \frac{1}{2}\right) |\uparrow\downarrow\rangle = \emptyset\, |\uparrow\downarrow\rangle \qquad m_j = 0$$
$$J_z |\downarrow\uparrow\rangle = \hbar \left(-\frac{1}{2} + \frac{1}{2}\right) |\downarrow\uparrow\rangle = \emptyset\, |\downarrow\uparrow\rangle \qquad m_j = 0$$
$$J_z |\downarrow\downarrow\rangle = \hbar \left(-\frac{1}{2} - \frac{1}{2}\right) |\downarrow\downarrow\rangle = -\hbar |\downarrow\downarrow\rangle \qquad m_j = -1$$

How about $J^2 = (\vec{S}^{(1)} + \vec{S}^{(2)})^2$? Let us just do the math:

$$J^2 |\uparrow\uparrow\rangle = (S^{(1)2} + S^{(2)2} + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)}) |\uparrow\uparrow\rangle =$$
$$= S^{(1)2} |\uparrow\uparrow\rangle + S^{(2)2} |\uparrow\uparrow\rangle + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)} |\uparrow\uparrow\rangle =$$

$$= \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar}{4}|\uparrow\uparrow\rangle + 2S_x^{(1)}S_x^{(2)}|\uparrow\uparrow\rangle +$$
$$+ 2S_y^{(1)}S_y^{(2)}|\uparrow\uparrow\rangle + 2S_z^{(1)}S_z^{(2)}|\uparrow\uparrow\rangle$$

Let's do these pieces one by one:

$$S_z^{(1)}S_z^{(2)}|\uparrow\uparrow\rangle = \frac{\hbar}{2}\cdot\frac{\hbar}{2}|\uparrow\uparrow\rangle = \frac{\hbar^2}{4}|\uparrow\uparrow\rangle$$

$S_z^{(1)}$ applied to $|\uparrow\uparrow\rangle$

Remember Pauli matrices. $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$: it flips $\uparrow$ into $\downarrow$

$$\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = \downarrow$$

Hence:

$$S_x^{(1)}S_x^{(2)}|\uparrow\uparrow\rangle = \frac{\hbar}{2}\cdot\frac{\hbar}{2}|\downarrow\downarrow\rangle = \frac{\hbar^2}{4}|\downarrow\downarrow\rangle$$

Similarly:

$$S_y^{(1)}S_y^{(2)}|\uparrow\uparrow\rangle = \left(\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{2}\right)|\downarrow\downarrow\rangle = -\frac{\hbar^2}{4}|\downarrow\downarrow\rangle$$

Putting it all together:

$$J^2|\uparrow\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \cancel{\frac{2\hbar^2}{4}|\downarrow\downarrow\rangle} - \cancel{\frac{2\hbar^2}{4}|\downarrow\downarrow\rangle} + \frac{2\hbar^2}{4}|\uparrow\uparrow\rangle$$
$$= 2\hbar^2|\uparrow\uparrow\rangle$$

Ah: $|\uparrow\uparrow\rangle$ also happens to be already an eigenstate of J^2 too!
with eigenvalue $\hbar^2 j(j+1)$ with $j = 1$.

Exercise: repeat with $|\downarrow\downarrow\rangle$ and find same result: $|\downarrow\downarrow\rangle$ is also an eigenstate of J^2 with eigenvalue $\hbar\, j(j+1)$ with $j = 1$.

The more interesting cases are $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. Let's see:

$$J^2|\uparrow\downarrow\rangle = \underbrace{\frac{3\hbar^2}{4}}_{S^{(1)2}}|\uparrow\downarrow\rangle + \underbrace{\frac{3\hbar^2}{4}}_{S^{(2)2}}|\uparrow\downarrow\rangle + 2\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)|\downarrow\uparrow\rangle +$$

$S_x^{(1)}$ flips particle 1 $\qquad$ $S_x^{(2)}$ flips particle 2

$$+2\underbrace{\left(+\frac{i\hbar}{2}\right)\left(-\frac{i\hbar}{2}\right)}_{\text{careful with sign}}|\downarrow\uparrow\rangle + 2\underbrace{\left(\frac{\hbar}{2}\right)}_{S_z^{(1)}:\,+\hbar/2}\left(-\frac{\hbar}{2}\right)|\uparrow\downarrow\rangle =$$

careful with sign:

$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$: $\quad S_y\uparrow = \frac{i\hbar}{2}\downarrow$

$\qquad S_y\downarrow = -\frac{i\hbar}{2}\uparrow$

particle-2 is spin down: $S_z^{(2)}\downarrow = -\frac{\hbar}{2}\downarrow$

$$J^2|\uparrow\downarrow\rangle = \hbar^2|\uparrow\downarrow\rangle + \hbar^2|\downarrow\uparrow\rangle$$: $|\uparrow\downarrow\rangle$ NOT an eigenstate of J^2!

$$J^2|\downarrow\uparrow\rangle = \hbar^2|\downarrow\uparrow\rangle + \hbar^2|\uparrow\downarrow\rangle$$: $|\downarrow\uparrow\rangle$ NOT an eigenstate of J^2 either!

What if I add the two expressions above? I get:

$$J^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = 2\hbar^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

and if I subtract them:

$$J^2\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \emptyset$$

This means that:

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ is an eigenstate of J^2 with eigenvalue $\hbar^2 j(j+1)$ with $j=1$

$\sqrt{2}$ ← normalization

I can easily verify that it is also an eigenstate of J_z with eigenvalue $\hbar m_j$; $m_j = 0$

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ is an eigenstate of J^2 with eigenvalue $\hbar j(j+1)$ with $j=0$ and also of J_z with eigenvalue $\hbar m_j = 0$.

I do not need more: I started with four eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$, and now I have four eigenstates of J^2 and J_z:

	m_1	m_2			j	m_j
$\|\uparrow\uparrow\rangle$	$1/2$	$1/2$		$\|1\ 1\rangle$	1	1
$\|\uparrow\downarrow\rangle$	$1/2$	$-1/2$		$\|1\ 0\rangle$	1	0
$\|\downarrow\uparrow\rangle$	$-1/2$	$1/2$	$\Longrightarrow$	$\|1\ {-1}\rangle$	1	-1
$\|\downarrow\downarrow\rangle$	$-1/2$	$-1/2$		$\|0\ 0\rangle$	0	0

where: $|1\ 1\rangle = |\uparrow\uparrow\rangle$

$$|1\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|1\ {-1}\rangle = |\downarrow\downarrow\rangle$$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

In general, if I have two particles of spin s_1 and s_2, their combination can have spin s between s_1+s_2 and $|s_1-s_2|$. Also, I can write the state $|s\ m\rangle$, an eigenstate of $S^2 = |\vec{S}_1+\vec{S}_2|^2$ and of $S_z = S_{1z} + S_{2z}$, as a linear combination of eigenstates of S_{1z} and S_{2z} (s_1 and s_2 are fixed: I choose to combine two states with spin s_1 and s_2):

$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m}\, |s_1\ m_1\rangle |s_2\ m_2\rangle$$

The $C^{s_1 s_2 s}_{m_1 m_2 m}$ coefficients are called <u>Clebsch-Gordan coefficients</u>.

E.g.: spin-1/2 and spin-1/2 combination:

$$C^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}\frac{1}{2}1} = 1 \qquad C^{\frac{1}{2}\frac{1}{2}1}_{-\frac{1}{2}-\frac{1}{2}-1} = 1$$

$$C^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}-\frac{1}{2}0} = \frac{1}{\sqrt{2}} \qquad C^{\frac{1}{2}\frac{1}{2}0}_{\frac{1}{2}-\frac{1}{2}0} = \frac{1}{\sqrt{2}}$$

$$C^{\frac{1}{2}\frac{1}{2}1}_{-\frac{1}{2}\frac{1}{2}0} = \frac{1}{\sqrt{2}} \qquad C^{\frac{1}{2}\frac{1}{2}0}_{-\frac{1}{2}\frac{1}{2}0} = -\frac{1}{\sqrt{2}}$$