(MORE ON) COMBINATION OF SPIN STATES

In a nutshell, the problem we want to solve is the following:
Let us imagine to have <u>two</u> particles with spin states χ_1 and χ_2.

We can indicate such a state in different, equivalent ways. Let us start from eigenstates of $S^{(1)}$ and $S^{(2)}$:

<u>spinor notation</u>

$$\chi_1 = a\,\chi_{1+} + b\,\chi_{1-} \qquad \text{where:} \qquad S_z^{(1)}\chi_{1+} = \frac{\hbar}{2}\chi_{1+}$$

simple case: $S = 1/2$; only two eigenstates are needed: $S_z = +1/2$ and $S_z = -1/2$ (spin-up and spin-down)

$$S_z^{(1)}\chi_{1-} = -\frac{\hbar}{2}\chi_{1-}$$

$$S^{2(1)}\chi_{1+} = \frac{3\hbar^2}{4}\chi_{1+}$$

$$S^{2(1)}\chi_{1-} = \frac{3\hbar^2}{4}\chi_{1-}$$

The two-particle system will be expressed by $\underline{\underline{\chi_1\chi_2}}$

E.g.:

$$\underline{\underline{S_x^{(1)}\chi_1\chi_2}} = \left(S_x^{(1)}\chi_1\right)\chi_2 = \left(S_x^{(1)}a\,\chi_{1+} + S_x^{(1)}b\,\chi_{1-}\right)\chi_2$$

$$= a\left[\left(S_x^{(1)}\chi_{1+}\right) + b\left(S_x^{(1)}\chi_{1-}\right)\right]\chi_2 =$$

$$= \underline{\underline{\frac{\hbar a}{2}\chi_{1-}\chi_2 + \frac{\hbar b}{2}\chi_{1+}\chi_2}}$$

<u>Dirac notation</u>

$$|\chi_1\rangle = a\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle + b\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \qquad \text{where:} \qquad S_z^{(1)}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \frac{\hbar}{2}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

$$= a\,|\uparrow\rangle + b\,|\downarrow\rangle \qquad\qquad S_z^{(1)}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = -\frac{\hbar}{2}\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$\left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle$ are eigenstates of S^2 and S_z:

$$S^{2(1)}\left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle = \frac{3\hbar^2}{4}\left|\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle$$

let me also indicate them as $|s\ m\rangle$

where $s = \frac{1}{2}$ and $m = \pm\frac{1}{2}$

The two-particle system will be expressed by:

$$|\chi_1\rangle|\chi_2\rangle = \sum_{m_1=-s_1}^{m_1=s_1}\sum_{m_2=-s_2}^{m_2=s_2} a_{m_1 m_2} |s_1\ m_1\rangle|s_2\ m_2\rangle$$

In simpler words: $|\chi_1\rangle|\chi_2\rangle$ can be written as a linear combination of two-particle states in which each particle is in an eigenstate of $S_z^{(1)}/S_z^{(2)}$ and $S^{2(1)}/S^{2(2)}$

E.g.: if both particles are in a pure spin-up state, then:

$$|\chi_1\rangle|\chi_2\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |\uparrow\rangle|\uparrow\rangle \underset{\text{in notation of previous lecture}}{=} |\uparrow\uparrow\rangle$$

E.g.:

$$\underline{\underline{S_y^{(2)}|\uparrow\uparrow\rangle}} = S_y^{(2)}|\uparrow\rangle|\uparrow\rangle \overset{S^{(2)}\text{ does nothing on particle 1}}{=} |\uparrow\rangle S_y^{(2)}|\uparrow\rangle =$$

$$= |\uparrow\rangle\cdot\frac{\hbar i}{2}|\downarrow\rangle = \underline{\underline{\frac{\hbar i}{2}|\uparrow\downarrow\rangle}}$$

Using Dirac's notation $|s\ m_s\rangle$:

$$S_y^{(2)}|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \frac{\hbar i}{2}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

<u>Linear representation</u>

generic spinor: $\begin{pmatrix} a \\ b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$
with spin $=\frac{1}{2}$

$\begin{pmatrix}0\\1\end{pmatrix}$: eigenstate of S^2 with eigenvalue $\frac{3\hbar^2}{4}$ and S_z with eigenvalue $-\frac{\hbar}{2}$

$\begin{pmatrix}1\\0\end{pmatrix}$: eigenstate of S^2 with eigenvalue $3\hbar^2/4$ and S_z with eigenvalue $\frac{\hbar}{2}$

$$S^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$S_{x,y,z} = \frac{\hbar}{2}\cdot\sigma_{x,y,z}$$

$\sigma_{x,y,z}$ Pauli matrices

The two-particle system will be expressed by a 4-dimensional vector obtained with direct product:

$$\chi_1 \otimes \chi_2$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

$$S_x^{(1)} \Rightarrow S_x^{(1)} \otimes I = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_x^{(2)} \Rightarrow I \otimes S_x^{(2)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S^{2(1)} \Rightarrow S^{2(1)} \otimes I = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (\text{do the math}) = I \otimes S^{2(2)}$$

This makes sense: I am using eigenstates of $S^{2(1)}$ and $S^{2(2)}$ as a basis for describing generic spinors $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}$.

Summary of notations:

English	spinor	Dirac	4-d linear
both particles are in spin-up state	$\chi_{1+}\,\chi_{2+}$	$\lvert\frac{1}{2}\frac{1}{2}\rangle\lvert\frac{1}{2}\frac{1}{2}\rangle$ or $\lvert\uparrow\uparrow\rangle$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
particle-1 in spin-up particle-2 in spin-down	$\chi_{1+}\,\chi_{2-}$	$\lvert\frac{1}{2}\frac{1}{2}\rangle\lvert\frac{1}{2}\,{-\frac{1}{2}}\rangle$ or $\lvert\uparrow\downarrow\rangle$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
particle-1 in spin-down particle-2 in spin-up	$\chi_{1-}\,\chi_{2+}$	$\lvert\uparrow\downarrow\rangle$	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
both particles in spin-down state	$\chi_{1-}\,\chi_{2-}$	$\lvert\downarrow\downarrow\rangle$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

GENERIC 2-particle STATE is a linear combination of the four states above.

We realize that we can describe any generic 2-particle state using the quantum numbers S_{z_1} and S_{z_2} : $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ or, in linear notation, $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, can be used as a basis for all generic states.

Let us define the <u>total</u> momentum operator $\vec{J} = \vec{S}^{(1)} + \vec{S}^{(2)}$.

The question we ask ourselves is: can We instead use, as a basis for generic states, eigenstates of the J^2 operator ? What would another good quantum number be ?

An easy guess is $J_z = S_z^{(1)} + S_z^{(2)}$. Let us measure J_z on our basis:

<u>Dirac notation</u>

$$J_z|\uparrow\uparrow\rangle = J_z\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left(S_z^{(1)} + S_z^{(2)}\right)\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle =$$

$$= \left(S_z^{(1)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left(S_z^{(2)}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right) =$$

$$= \frac{\hbar}{2}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \frac{\hbar}{2}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \hbar|\uparrow\uparrow\rangle$$

<u>spinor notation</u>

if χ_1 is a state for particle 1 which is an eigenstate of $S_z^{(1)}$ with eigenvalue $\hbar m_1$, and χ_2 is a particle-2 state which is an eigenstate of $S_z^{(2)}$ with eigenvalue $\hbar m_2$, then:

$$J_z\chi_1\chi_2 = \left(S_z^{(1)} + S_z^{(2)}\right)\chi_1\chi_2 = \left(S_z^{(1)}\chi_1\right)\chi_2 + \chi_1\left(S_z^{(2)}\chi_2\right) =$$

$$= \hbar(m_1 + m_2)\chi_1\chi_2$$

<u>$\chi_1\chi_2$</u> is eigenstate of J_z, with eigenvalue $\hbar(m_1+m_2)$

Let us look at J^2 in $\underset{4d}{\wedge}$ linear representation:

$$J^2 = (\vec{S}^{(1)} + \vec{S}^{(2)})^2 = S^{2(1)} + S^{2(2)} + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)} =$$

$$= (S_x^{(1)} \otimes I)^2 + (S_y^{(1)} \otimes I)^2 + (S_z^{(1)} \otimes I)^2 +$$

$$+ (I \otimes S_x^{(2)})^2 + (I \otimes S_y^{(2)})^2 + (I \otimes S_z^{(2)})^2 +$$

$$+ 2\, S_x^{(1)} \otimes S_x^{(2)} + 2\, S_y^{(1)} \otimes S_y^{(2)} + 2\, S_z^{(1)} \otimes S_z^{(2)}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S^{(1),(2)}_{\substack{x\\y\\z}}$ are $\frac{\hbar}{2} \cdot \sigma_{\substack{x\\y\\z}}$, Pauli matrices.

J^2, using the rules for $\otimes$ defined on the second page, becomes a 4×4 matrix. We can diagonalize it, obtaining these 4 eigenvectors:

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ with eigenvalue $\hbar^2 \cdot 1(1+1) = 2\hbar^2$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ with eigenvalue $\hbar^2 1(1+1) = 2\hbar^2$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ with eigenvalue $\hbar^2\ 1(1+1) = 2\hbar^2$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ with eigenvalue $\hbar^2 \cdot 0(0+1) = 0$

What is the value of $J_z = S_z^{(1)} + S_z^{(2)}$ on these <u>four</u> states? From top to bottom, I find that they are, besides being eigenstates of J^2, also eigenstates of J_z, with eigenvalues $1, 0, -1$, and 0, respectively.

Let me call these four states $|\,j \ \ m_j\,\rangle$, we have:

$$\left|\overset{j}{1}\ \ \overset{m_j}{1}\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \left|\overset{s^{(1)}}{\tfrac{1}{2}}\ \overset{s_z^{(1)}}{\tfrac{1}{2}}\right\rangle \left|\overset{s^{(2)}}{\tfrac{1}{2}}\ \overset{s_z^{(2)}}{\tfrac{1}{2}}\right\rangle$$

Dirac   linear notation   again, Dirac

$$|\,1\ \ 0\,\rangle = \begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right)$$

$$|\,1\ \ -1\,\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle$$

$$|\,0\ \ 0\,\rangle = \begin{pmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\left|\frac{1}{2}\ \frac{-1}{2}\right\rangle - \left|\frac{1}{2}\ \frac{-1}{2}\right\rangle\left|\frac{1}{2}\ \frac{1}{2}\right\rangle\right)$$

Conclusion: I can describe a 2-particle system using a linear combination of eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$, with eigenvalues m_1 and m_2, OR a linear combination of eigenstates of J^2 and J_z, with eigenvalues

$\hbar^2 j(j+1)$ where $j = s^{(1)} + s^{(2)}, \dots, |s^{(1)} - s^{(2)}|$ and

$\hbar m_j$ where $m_j = m_1 + m_2$ (m_j is fixed by m_1 and m_2)

Formula:

$$|\,j \ \ m_j\,\rangle = \sum_{m_1+m_2=m_j} C^{s_1 s_2 j}_{m_1 m_2 m_j}\, |\,s_1\ m_1\,\rangle\,|\,s_2\ m_2\,\rangle$$

$C^{s_1 s_2 j}_{m_1 m_2 m_j}$ are the Clebsch-Gordan coefficients