ANGULAR MOMENTA ADDITION

We expect to see, when H atom is in magnetic field, that emission spectrum lines split into sets of 3. Experimentally, we instead see splits in <u>even</u> numbers (when $\vec{B}$ field is weak). The problem is that we forgot the spin interaction with $\vec{B}$. It is reasonable to think that I should look at the total angular momentum: orbital and intrinsic. How can we describe the addition of angular momenta?

In the case of spin-1/2, we saw we can use a 2×2 linear representation in which the $\vec{S}$ components are proportional ($\hbar/2$ is the factor) to the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenstates of S^2 and S_z, with eigenvalues $\hbar^2 s(s+1) = 3\hbar/4$ and $\hbar m = \pm\hbar/2$ (remember: spin = 1/2)

We can generalize Pauli matrices to higher dimensions:

$$S_z = \hbar \begin{pmatrix} s & 0 & 0 & \dots & & 0 \\ 0 & s-1 & 0 & & & \vdots \\ \vdots & 0 & s-2 & & & \vdots \\ \vdots & & & \ddots & & \\ & & & & -s+1 & 0 \\ 0 & 0 & \dots & & 0 & -s \end{pmatrix} \quad S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & b_s & 0 & \dots & & 0 \\ b_s & 0 & b_{s-1} & 0 & & \vdots \\ 0 & b_{s-1} & 0 & b_{s-2} & & \vdots \\ \vdots & 0 & b_{s-2} & 0 & & \\ & & & & 0 & b_{-s+1} \\ 0 & \dots & & & b_{-s+1} & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & b_s & & \dots & & 0 \\ -b_s & 0 & b_{s-1} & & & \vdots \\ 0 & -b_{s-1} & 0 & & & \vdots \\ \vdots & & & \ddots & & \\ & & & & & b_{-s+1} \\ 0 & & & & -b_{-s+1} & 0 \end{pmatrix}$$

where $b_j = \sqrt{(s+j)(s+1-j)}$

<u>$\ell = 1$</u> case:

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Let us start with simplest non-trivial case: spin 1/2 + spin 1/2.

I ask myself what is the total momentum $\vec{J} = \vec{S}_1 + \vec{S}_2$.

Let us build a 4-dimensional space from two 2-dimensional sources. Intuitively, I build combinations of spin-up and spin-down states:

$$\overset{S(1)\text{ part}}{\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}} \otimes \overset{S(2)\text{ part}}{\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}} = \begin{pmatrix}\uparrow\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\\ \downarrow\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\uparrow\ \uparrow\\ \uparrow\ \downarrow\\ \downarrow\ \uparrow\\ \downarrow\ \downarrow\end{pmatrix}$$

spin-up component

spin-down component

Same operation on operators (with some interesting results):

$S^{(1)}_{x,y,z} \otimes I_2$: apply $S_{x,y,z}$ on (1) part, and leave alone (2) part

$$S_x^{(1)} \otimes I_2 = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

$$I_2 \otimes S_x^{(2)} = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

Since $S^{(1)}$ operates only on the first spin state, and $S^{(2)}$ only on the second, we also have that:

$$S_x^{(1)} S_x^{(2)} = (S_x^{(1)} \otimes I_2)(I_2 \otimes S_x^{(2)}) = S_x^{(1)} \otimes S_x^{(2)}$$

This means the following:

$$\underbrace{\left(S_z^{(1)} \otimes S_z^{(2)}\right)}_{\text{honest 4x4 operator}} \left(\overset{(1)}{\begin{pmatrix}\uparrow\\0\end{pmatrix}} \otimes \overset{(2)}{\begin{pmatrix}\uparrow\\0\end{pmatrix}}\right) = S_z^{(1)}\overset{(1)}{\begin{pmatrix}1\\0\end{pmatrix}} \otimes S_z^{(2)}\overset{(2)}{\begin{pmatrix}1\\0\end{pmatrix}} = \frac{\hbar^2}{4}\overset{(1)}{\begin{pmatrix}\uparrow\\0\end{pmatrix}} \otimes \overset{(2)}{\begin{pmatrix}\uparrow\\0\end{pmatrix}}$$

applied to state in which both spins are up

Let us go back to $\vec{J} = \vec{S}^{(1)} + \vec{S}^{(2)}$. Now I can write:

$$J^2 = (\vec{S}^{(1)})^2 + (\vec{S}^{(2)})^2 + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)} =$$

$$= (S_x^{(1)} \otimes I_2)^2 + (S_y^{(1)} \otimes I_2)^2 + (S_z^{(1)} \otimes I_2)^2 +$$

$$+ (I_2 \otimes S_x^{(2)})^2 + (I_2 \otimes S_y^{(2)})^2 + (I_2 \otimes S_z^{(2)})^2 +$$

$$+ 2 (S_x^{(1)} \otimes S_x^{(2)} + S_y^{(1)} \otimes S_y^{(2)} + S_z^{(1)} \otimes S_z^{(2)})$$

note: e.g. $|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ according to definition in previous page

I can diagonalize this matrix and find $\underline{4}$ eigenstates:

<u>singlet</u> $\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$ eigenvalue: $0(0+1) = 0$

<u>triplet</u> $\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$; $|\uparrow\uparrow\rangle$; $|\downarrow\downarrow\rangle$ eigenvalue: $2 = 1(1+1)$

How about J_z? $J_z = S_z^{(1)} + S_z^{(2)}$ and let us take two spinors, one per subspace, with eigenvalues of $S_z^{(i)}$ equal to m_i. I.e., we have

$$S_z^{(1)} \chi_1 = \hbar m_1 \chi_1 \quad \text{and} \quad S_z^{(2)} \chi_2 = \hbar m_2 \chi_2$$

We have:

$$J_z (\chi_1 \otimes \chi_2) = (S_z^{(1)} + S_z^{(2)})(\chi_1 \otimes \chi_2) = (S_z^{(1)} \chi_1 \otimes I_2 \chi_2) +$$

$$+ (I_2 \chi_1 \otimes S_z^{(2)} \chi_2) = \hbar m_1 (\chi_1 \otimes \chi_2) + \hbar m_2 (\chi_1 \otimes \chi_2) =$$

$$= \hbar (m_1 + m_2) (\chi_1 \otimes \chi_2)$$

I can now fully define the states I found above:

singlet: $|\overset{j}{0} \ \overset{m_j}{0}\rangle = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)$

triplet: $|\overset{j}{1} \ \overset{m_j}{1}\rangle = |\uparrow\uparrow\rangle$ $\quad |\overset{j}{1} \ \overset{m_j}{-1}\rangle = |\downarrow\downarrow\rangle$ $\quad |\overset{j}{1} \ \overset{m_j}{0}\rangle = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$

My space is 4-dimensional, I need 4 eigenstates to span it. I can choose:

$\overset{(1)(2)}{\vert\uparrow\uparrow\rangle}$	$S^{(1)}=\frac{1}{2}$	$S_z^{(1)}=\frac{1}{2}$	$S^{(2)}=\frac{1}{2}$	$S_z^{(2)}=\frac{1}{2}$	($\hbar=1$...)
$\vert\uparrow\downarrow\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	
$\vert\downarrow\uparrow\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	
$\vert\downarrow\downarrow\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	

Alternatively, I can choose eigenstates of $J^2=(S_1^{(1)}+S^{(2)})^2$ and $J_z=S_z^{(1)}+S_z^{(2)}$

$\overset{j\ m_j}{\vert 0\ 0\rangle}$	$j=0$	$m_j=0$
$\vert 1\ 1\rangle$	1	1
$\vert 1\ 0\rangle$	1	0
$\vert 1\ -1\rangle$	1	-1

Relation among the two sets:

$$|0\ 0\rangle=\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle-|\uparrow\downarrow\rangle\right)$$

$$|1\ 1\rangle=|\uparrow\uparrow\rangle$$

$$|1\ 0\rangle=\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle\right)$$

$$|1\ -1\rangle=|\downarrow\downarrow\rangle$$

NOTE: when I combine two states with momentum $S^{(1)}$ and $S^{(2)}$, the total angular momentum can assume values between $S^{(1)}+S^{(2)}$ and $|S^{(1)}-S^{(2)}|$, while m_j will assume values between $-j$ and $+j$.

CONCLUSION: the combination of two states $|l_1\ m_1\rangle$ and $|l_2\ m_2\rangle$ can be written as a linear combination of states $|j\ m_j\rangle$ with $j=l_1+l_2,\ l_1+l_2-1,\ldots|l_1-l_2|$ and $m_j=m_1+m_2$ (many j's, but all states with the <u>same</u> m_j)

PDG

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J M	J M	...
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

1/2×1/2

m_1	m_2	1 +1
+1/2	+1/2	1

m_1	m_2	1 0	0 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1 −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2 +3/2
+1	+1/2	1

m_1	m_2	3/2 +1/2	1/2 +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2 −1/2	1/2 −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2 −3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2 +5/2
+2	+1/2	1

m_1	m_2	5/2 +3/2	3/2 +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2 +1/2	3/2 +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2 −1/2	3/2 −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2 −3/2	3/2 −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2 −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2 +2
+3/2	+1/2	1

m_1	m_2	2 +1	1 +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2 0	1 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2 −1	1 −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2 −2
−3/2	−1/2	1

2×1

m_1	m_2	3 +3
+2	+1	1

m_1	m_2	3 +2	2 +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3 +1	2 +1	1 +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3 0	2 0	1 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3 −1	2 −1	1 −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3 −2	2 −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3 −3
−2	−1	1

3/2×1

m_1	m_2	5/2 +5/2
+3/2	+1	1

m_1	m_2	5/2 +3/2	3/2 +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2 −3/2	3/2 −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2 −5/2
−3/2	−1	1

1×1

m_1	m_2	2 +2
+1	+1	1

m_1	m_2	2 +1	1 +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2 0	1 0	0 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2 −1	1 −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2 −2
−1	−1	1