# FINAL REVIEW

QUANTUM STATISTICAL MECHANICS

Let us consider a system of many particles, each of which could have energy $E_s$ (s indicates the energy level, each of which has degeneracy $g_s$).
Let us also assume that the system has total energy $E$, and a total number of particles $N$. Then, the average number of particles in each state with energy $E_s$ is:

$$P_s = \frac{n_s}{g_s} = e^{-(E_s-\mu)/k_BT}$$

$n_s$ ← # particles with energy $E_s$

$g_s$ ← degeneracy of energy level s

MAXWELL-BOLTZMANN
DISTINGUISHABLE PARTICLES

$$= \frac{1}{e^{(E_s-\mu)/k_BT}-1}$$

BOSE-EINSTEIN
IDENTICAL BOSONS

$$= \frac{1}{e^{(E_s-\mu)/k_BT}+1}$$

FERMI-DIRAC
IDENTICAL FERMIONS

$\mu$ = chemical potential; it is a function of $T$

Typical constraints:

$$N = \sum_s n_s$$

$$E = \sum_s E_s \cdot n_s$$

In some of our problems, we remove the first constraint ($N = \sum n_s$) and thus obtain that $\mu(T) = 0$. Also, our concern becomes finding what $g_s$ is, for the system at hand. Here are some examples:

1) 3D photons in a box: in the k-space, each state occupies a volume $\frac{\pi^3}{V}$. An octant shell at radius $k$ occupies the volume $\frac{1}{8} 4\pi k^2 dk$.

Since the energy of a single photon is $\hbar c k$, that shell is the locus of all states with the same energy. Hence, its volume divided by the volume of a single state gives the <u>degeneracy</u> of the energy state with energy $\hbar c k$ (note: we assume $V$ large enough that the states become dense in the k-space, and I can replace the discrete Es with the continuous $E_K = \hbar c k$).

Finally:

discrete, continuous; POLARIZATION of photons

$$g_s \rightarrow g(k)dk = \frac{1}{8}\frac{4\pi k^2 dk}{\pi^3/V} \cdot 2 = \frac{V k^2}{\pi^2} dk$$

what is the probability of having a photon with energy $E_K$? Bosons → B-E:

$$p(E_K) = g(k) \cdot \frac{1}{e^{E_K/k_BT} - 1}$$

⇐ from here we can work in terms of $\omega = ck$ (remember the piece $g(\omega)d\omega = g(k)dk \Rightarrow$ need a factor $c$: $\frac{d\omega}{dk} = c$)

not really a probability but proportional to

2) 3D phonons in a crystal: again, we move to k-space:

$$g(k)dk = \underbrace{\frac{1}{8} 4\pi k^2 dk}_{\text{shell volume}} \cdot \frac{1}{\pi^3/V} \cdot 3$$

$\pi^3/V$: volume of 1 state; 3: POLARIZATIONS of phonons

this time, there is a max value of $k$ that can be achieved.

$$g(k)dk = g(\omega)d\omega \Rightarrow g(\omega) = g(k(\omega))\frac{dk}{d\omega} = \frac{3}{2}V\frac{\omega^2}{\pi^2 v^3}$$

(dispersion relation: $\omega = v \cdot k$)

$$U = \text{total energy in phonons} = \int_0^{\omega_{max}} g(\omega) \cdot \frac{1}{e^{\hbar\omega/k_BT} - 1} \cdot \hbar\omega \, d\omega$$

$g(\omega)$: density of states; $\frac{1}{e^{\hbar\omega/k_BT}-1}$: B-E probability distribution; $\hbar\omega$: energy of 1 state

Since there exist a $\omega_{max}$, I have a finite number of possible <u>states</u>. Let us call this 3N (not to be confused with a number of phonons: since we set $\mu = 0$, the number of phonons is not constrained).

$$3N = \int_0^{\omega_{max}} g(\omega)\, d\omega$$ : provides relation between $v$, $\omega_{max}$, $N$.

3) 3D Fermi gas of electrons: → in this case, $\mu \neq 0$: the number of electrons is fixed, while photons and phonons can be created and destroyed

spin-up and spin-down

$$g(k)\,dk = \frac{1}{8} 4\pi k^2 dk \cdot \frac{1}{\pi^3/V} \cdot 2$$

IF $T=0$

{ all electrons occupy, in ground state, the lowest possible energies, but cannot use same state twice. In k-space, they occupy an octant of radius $E_F$ }

$$N = \text{number of electrons} = \int_0^{k_F} g(k) \cdot \frac{1}{e^{(E_k-\mu)/k_BT}+1}\, dk$$

F-D distribution

$$E_F = \frac{\hbar^2 k_F^2}{2m} \quad \text{Fermi energy}$$

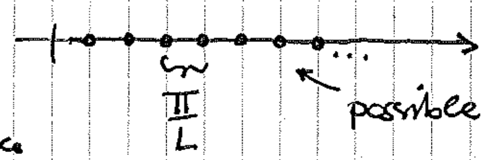
$$E = \text{total energy} = \int_0^{k_F} g(k) \cdot \frac{1}{e^{(E_k-\mu)/k_BT}+1} \cdot \frac{\hbar^2 k^2}{2m}\, dk$$

Equations for $N$ and $E$ allow me to find formulae for $k_F$ and $\mu(T)$ as a function of $n$ (= density of electrons, $N/V$), $T$ and $E$.

Interesting note: $\mu(0) = E_F$ ( $\phi(E) = 0$ if $E > E_F$, and $=1$ if $E < E_F$, when $T=0$, hence $\mu(0) = E_F$ )

yes, it's OK

4) 1D Fermi gas of electrons:




$$g(k)\,dk = 2 \frac{dk}{\pi/L}$$

infinitesimal length in k-space

1-D volume occupied by one state

spin-up or spin-down

T=0 CASE:

$$N = \text{number of electrons on wire} = \frac{2L}{\pi} \cdot k_F = \int_0^{k_F} \frac{2L}{\pi}\, dk$$

$$k_F = \frac{\pi n}{2}$$ linear density of electrons

$$E_F = \text{Fermi energy} = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 \pi^2}{8m} \cdot n^2$$

linear density of electrons, do not confuse with particle in quantum well...

# WKB APPROXIMATION

The really fast summary of WKB techniques:

1) $E > V(x)$:

$$\psi(x) \simeq \frac{C_+}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int^x p(x')dx'} + \frac{C_-}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int^x p(x')dx'}$$

$$\simeq \frac{1}{\sqrt{p(x)}} \left\{ C_s \sin\left(\frac{1}{\hbar}\int^x p(x')dx'\right) + C_c \cos\left(\frac{1}{\hbar}\int^x p(x')dx'\right)\right\}$$

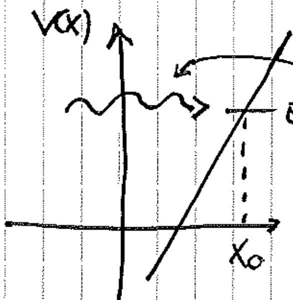
where $p(x) = \sqrt{2m(E - V(x))}$

2) $E < V(x)$:

$$\psi(x) \simeq \frac{C_+}{\sqrt{|p(x)|}} e^{\frac{1}{\hbar}\int^x |p(x')| dx'} + \frac{C_-}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int^x |p(x')| dx'}$$

note: $|p(x')|$ modulus

3) connection formulae:

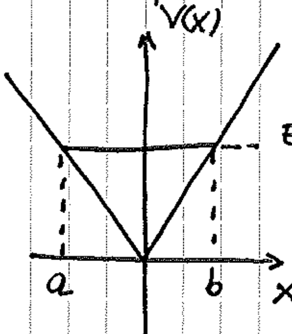



classical turning point

$$\psi(x) \simeq \frac{1}{\sqrt{p(x)}} C_s \sin\left(\frac{1}{\hbar}\int_x^{x_0} p(x')dx' + \frac{\pi}{4}\right)$$

add $\frac{\pi}{4}$ for each soft wall (soft wall: $\frac{dV}{dx} \neq \infty$)

Example:

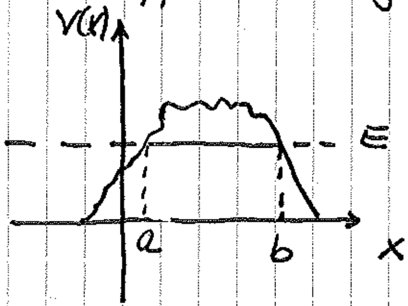



quantization condition:

$$\frac{1}{\hbar}\int_a^b \sqrt{2m(E - V(x))}\, dx + 2 \cdot \frac{\pi}{4} = n \cdot \pi$$

two soft walls

4) Approximate formula for tunnelling




$$T = \text{tunnelling probability} = e^{-2\gamma}$$

$$\gamma = \frac{1}{\hbar} \int_a^b \left| \sqrt{2m\,(E - V(x))} \right| dx$$

$E < V(x)$ : need modulus

"lifetime formula":

$$\tau = \frac{2r}{v} e^{2\gamma}$$

where $r$ is some scale of distance relevant in the problem (e.g., the nucleus radius in the Gamow's model of $\alpha$ decays) and $v$ is some velocity scale relevant in the problem (e.g., $v = \sqrt{\frac{k_B T}{m}}$, if we are considering our particle as moving due to thermal motion)

NOTE: WKB works in semi-classical limit: wavelength associated with state with energy $E$ is much smaller than the scale over which the potential $V(x)$ varies.

THE VARIATIONAL PRINCIPLE

$E_{\text{ground state}} \leq \langle \psi | H | \psi \rangle \quad \forall \psi$

one can obtain a better estimate of $E_{\text{ground state}}$ by making $\psi$ a function of some parameter $\lambda_i$, then minimize $\langle \psi | H | \psi \rangle$ as a function of $\lambda_i$:

$\frac{\partial \langle H \rangle}{\partial \lambda_i} = 0$ for some $\tilde{\lambda}_i$, then calculate $\langle H(\tilde{\lambda}_i) \rangle$

IMPORTANT: $\psi(\lambda_i)$ must be normalized: $\langle \psi(\lambda_i) | \psi(\lambda_i) \rangle = 1$. The normalization coefficient may depend on $\lambda_i$!