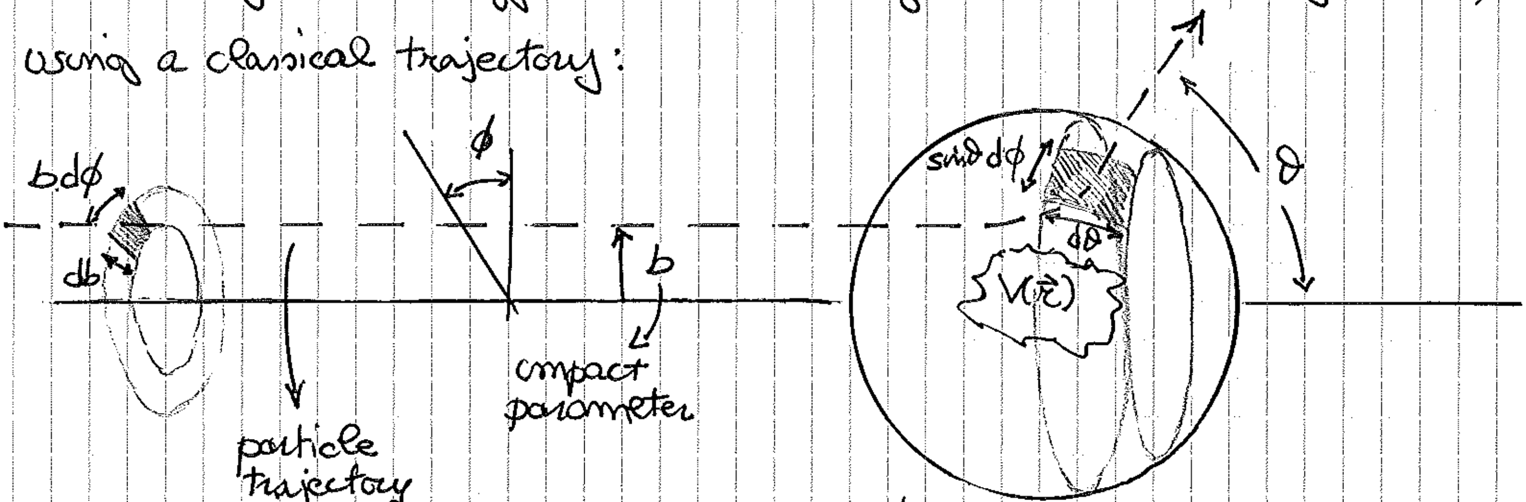
SCATTERING

While in 1-D we can describe the effect of localized potentials $V(x)$ in terms of reflection and transmission coefficients, the situation is a bit more complicate in 3-D, when particles can scatter at any angle. Let us define the differential scattering cross section as follows, using a classical trajectory:



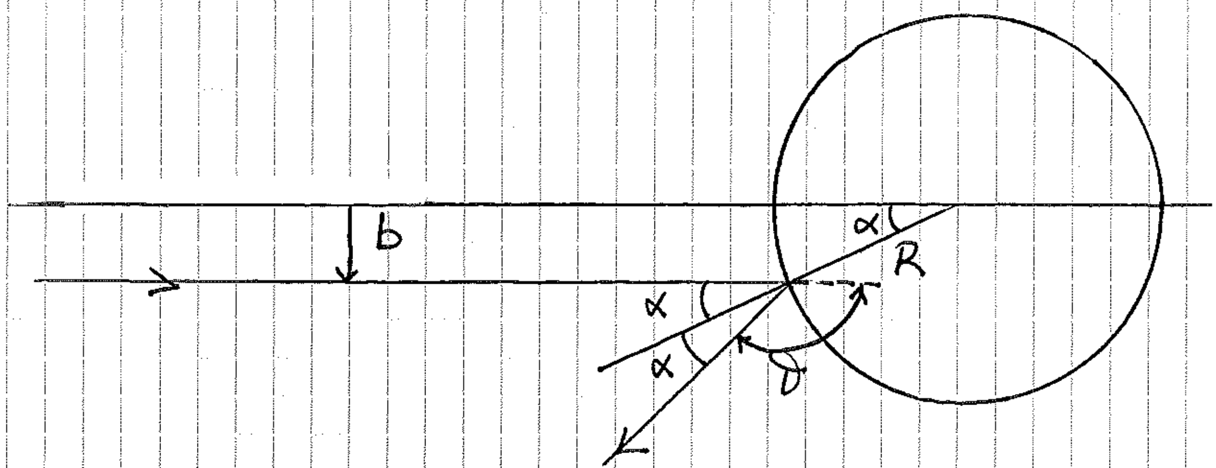
The differential scattering cross section $\frac{d\sigma}{d\Omega}$ tells us where, in θ and ϕ (in a system of polar coordinates centered where the scattering potential $V(\vec{r})$ is located), the particles coming from the infinitesimal area $b\,d\phi\,db$ are scattered by the potential $V(\vec{r})$. $d\sigma$ is the small area from which the particles originate, $d\Omega$ is the solid angle within which they are scattered. Clearly:

$$\frac{d\sigma}{d\Omega} = \frac{b\,d\phi\,db}{\sin\theta\,d\phi\,d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

we add absolute value to keep this positive (note that $db/d\theta$ is generally negative: the smaller b, the more head-on the collision with the scatterer, the larger θ)

My problem typically consists in finding $b(\theta)$. Typically, $V(\vec{r}) = V(r)$, and there is no dependence on ϕ (the problem is cylindrically symmetric). Let me call $d\sigma = D(\theta)\,d\Omega$, and find $D(\theta)$ in a classical problem: the hard sphere.

NOTE: $d\sigma$ has dimensions of an area, while $d\Omega$ is a solid angle



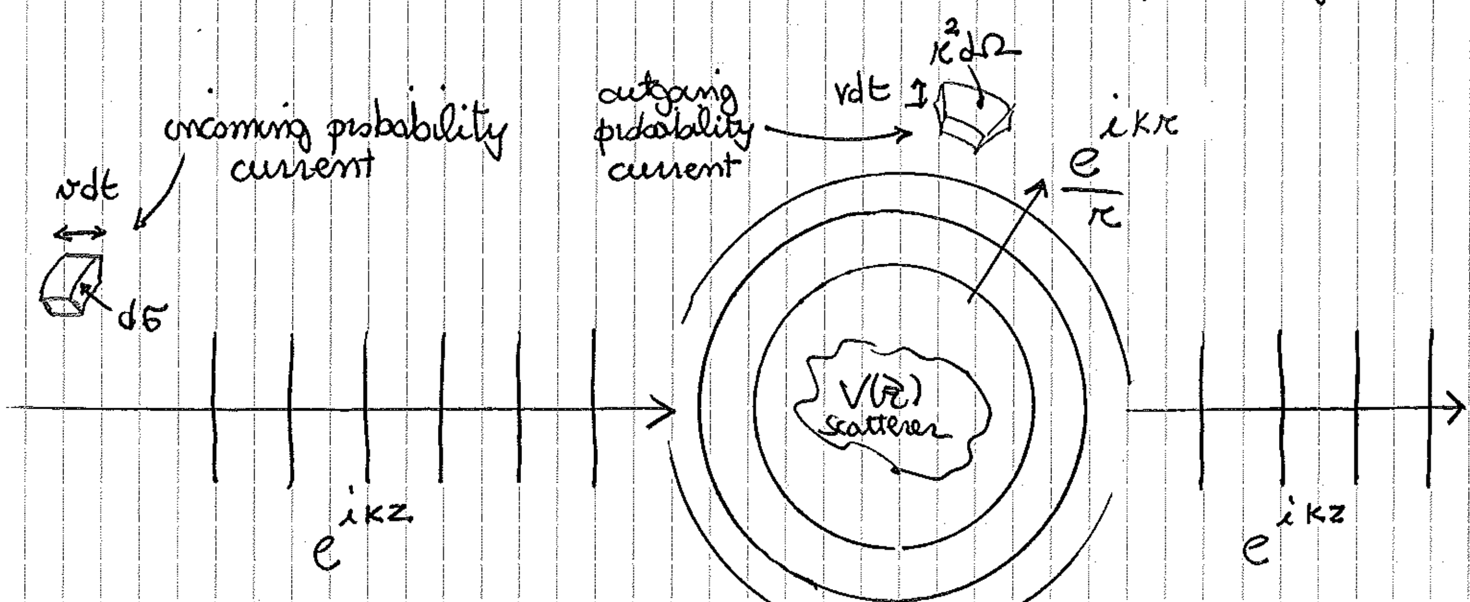
b = impact parameter = $R\sin\alpha$
ϑ = scattering angle = $\pi - 2\alpha$ $\Rightarrow$ $b = R\sin\left(\frac{\pi}{2} - \frac{\vartheta}{2}\right) = R\cos\left(\frac{\vartheta}{2}\right)$

$$D(\vartheta) = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right| = \frac{R\cos\left(\frac{\vartheta}{2}\right)}{\sin\vartheta}\cdot R\cdot\frac{1}{2}\sin\left(\frac{\vartheta}{2}\right) = \frac{R^2}{4}$$

$$\sigma = \text{Total cross section} = \int D(\vartheta)\,d\Omega = \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\,\underbrace{\int_0^{\pi} d\vartheta\,\sin\vartheta}_{2}\,\underbrace{D(\vartheta)}_{R^2/4} = \pi R^2$$

This is what we expect: the cross section represents the area occupied by the scatterer as "seen" by the particle. By "seeing", we mean that if the particle enters that area, its direction gets modified (it scatters in some direction). If we stare at a sphere of radius R, we indeed see an area πR^2 projected by the sphere.

The situation is similar in quantum mechanics: we look at probability currents.



The idea is that my incoming plane wave, $\propto e^{ikz}$, will produce an outgoing spherical wave upon encountering the potential $V(\vec{r})$. We hence look for solutions of Schroedinger's equation in the form:

$$\psi(r,\vartheta) \approx \psi_{\text{incoming}} + \psi_{\text{scattered}} = A\left(e^{ikz} + f(\vartheta)\frac{e^{ikr}}{r}\right)$$

valid for large r, away from the scattering potential. In that region, $k = \sqrt{2mE}/\hbar$, and all the info I need about $d\sigma/d\Omega$ is contained in $f(\vartheta)$:

$$dP = \text{probability of being in incoming volume } d\sigma \cdot v\,dt = |\psi_{\text{incoming}}|^2 \underset{\text{volume}}{dV} = |A|^2 v\,dt\,d\sigma$$

$$\underset{\text{must be}}{=} \text{probability of being in scattered volume } r^2 d\Omega = |\psi_{\text{scattered}}|^2 \underset{\text{volume}}{dV} = |A|^2 |f(\vartheta)|^2 v\,dt \cdot \underbrace{r^2 d\Omega}_{\substack{\text{area subtended}\\ \text{by solid angle}\\ d\Omega}} \cdot \underbrace{\frac{1}{r^2}}_{\substack{\text{spherical}\\ \text{wave function}\\ \propto \frac{1}{r}}}$$

$$\Longrightarrow \quad \frac{d\sigma}{d\Omega} = |f(\vartheta)|^2$$

Let us study a centrally symmetric potential, $V(r)$, i.e., no dependence on ϑ or φ. The radial Schroedinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}u = Eu$$

Let us now assume that $V(r)$ is localized stronger than $1/r^2$ (e.g., $1/r^3$; note: Coulomb potential, $\propto 1/r$, does NOT satisfy this condition).

Then, for large r, we can approximately write:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}u = Eu$$

The solutions to this equation that are outgoing waves are the spherical Hankel's functions of the first kind:

$$H_\ell^{(1)}(kr) \xrightarrow[\text{for large } r]{} \frac{(-i)^{\ell+1}}{kr} e^{ikr}$$

The general solution is a sum of solutions for all values of ℓ and m:

$$\psi(r,\theta,\phi) = A\left(\underbrace{e^{ikz}}_{\text{incoming wave}} + \underbrace{\sum_{\ell,m} c_{\ell m} H_\ell^{(1)}(kr)}_{\text{scattered wave}} \underbrace{Y_\ell^m(\theta,\phi)}_{\text{solution of Schroedinger's angular equation}} \right)$$

Since $V(r)$ is not depending on ϕ, and e^{ikz}, the incoming wave, is also not depending on ϕ, the problem has an azimuthal symmetry $\Rightarrow$ the solution cannot depend on ϕ, hence only terms with $m=0$ can appear in the sum.

Since:

$$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\ P_\ell(\cos\theta)$$ Legendre polynomial

and defining:

$$a_\ell = \text{partial wave amplitude} = c_{\ell 0}\,\frac{(-i)^{\ell+1}}{k\sqrt{4\pi(2\ell+1)}}$$

we can write, for <u>large r</u>:

$$\psi(r,\theta) = A\left(e^{ikz} + \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)\,\frac{e^{ikr}}{r}\right)$$

Comparing this expression with what we started with, we get:

$$f(\theta) = \sum_{\ell=0}^{\infty} a_\ell (2\ell+1) P_\ell(\cos\theta)$$

and:

$$\frac{d\sigma}{d\Omega} = D(\theta) = |f(\theta)|^2 = \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty} a_\ell a_{\ell'}^* (2\ell+1)(2\ell'+1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

We can simplify this expression using the orthogonality relation of Legendre polynomials:

$$\int P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\Omega = \frac{4\pi}{(2\ell+1)}\,\delta_{\ell\ell'}$$

and use it to find a compact expression for the total cross section:

$$\boxed{\sigma =} \int d\Omega \, D(\theta) = \int d\Omega \sum_{\ell} \sum_{\ell'} a_\ell a_{\ell'}^* (2\ell+1)(2\ell'+1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta) =$$

$$= \sum_{\ell} \sum_{\ell'} a_\ell a_{\ell'}^* (2\ell+1)(2\ell'+1) \int d\Omega \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) =$$

$$= \sum_{\ell} \sum_{\ell'} a_\ell a_{\ell'}^* (2\ell+1)(2\ell'+1) \frac{4\pi}{2\ell+1} \delta_{\ell\ell'} =$$

$$\boxed{= \sum_{\ell=0}^{\infty} |a_\ell|^2 \, 4\pi (2\ell+1)}$$

Our job is to find the partial wave amplitudes a_ℓ.

Let us look at the case of a quantum hard sphere:

$$V(r) = \begin{cases} 0 & r > R \\ \infty & r \leq R \end{cases}$$

We can use the following identity:

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \underbrace{j_\ell(kr)}_{\text{Bessel functions}} P_\ell(\cos\theta)$$

obtaining: (here we understand why the definition of a_ℓ kept a $(2\ell+1)$ term around)

$$\psi(r,\theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \left(j_\ell(kr) + ik a_\ell H_\ell^{(1)}(kr) \right) P_\ell(\cos\theta)$$

The boundary condition is:

$$\psi(R,\theta) = 0$$

Since $P_\ell(\cos\theta)$ are orthonormal, this means that each term in the sum has to be individually equal to 0. This gives us the value of a_ℓ!

We obtain:

$$a_\ell = \frac{i\, j_\ell(KR)}{K\, H_\ell^{(1)}(KR)}$$

This is still an exact solution. Let us look at what happens for low-energy scattering, i.e., K small and $KR \ll 1$:

$$j_\ell(KR) \approx \frac{2^\ell \ell!}{(2\ell+1)!} (KR)^\ell$$

$$H_\ell^{(1)}(KR) \approx -i \frac{(2\ell)!}{2^\ell \ell!} (KR)^{-\ell-1}$$

(compare with Griffiths table 4.4, noting that $H_\ell^{(1)}(KR) = j_\ell(KR) + i n(KR)$)

wins over $j_\ell(KR)$ for small KR

Finally:

$$a_\ell = \frac{-1}{K} \frac{1}{(2\ell+1)} \left(\frac{2^\ell \ell!}{(2\ell)!} \right)^2 (KR)^{2\ell+1}$$

Since $KR \ll 1$, the $\ell = 0$ term dominates:

$$\sigma = \sum_{\ell=0}^{\infty} (2\ell+1) 4\pi |a_\ell|^2 \simeq 4\pi |a_0|^2 = \frac{4\pi}{K^2} (KR)^2 = \underline{\underline{4\pi R^2}}$$

Instead of seeing only the projected area, in the quantum case the incoming wave sees the whole surface!

Interesting example: the total proton-proton cross section is 100mb = = 100 millibarn = 10 fm^2

The proton radius is 1 fm, and $4\pi r_{proton}^2 \simeq 10\ fm^2$!