THE AHARONOV-BOHM EFFECT (BERRY'S PHASE)

In classical electromagnetism, potentials (ϕ and $\vec{A}$) provide a convenient mean to solve equations, but only the fields ($\vec{E}$ and $\vec{B} \propto \vec{H}$) have physical effects. The Aharonov-Bohm effect shows that this is not the case, and that quantum mechanics explains an action-at-a-distance that classical EM is unable to describe. This effect was demonstrated in 1959, and it is an example of a Berry's phase. Here is Berry's phase equation:

$$\gamma_m(T) = i \oint \langle \psi_m | \nabla_R \psi_m \rangle \cdot dR$$

where T is the time it takes a time-dependent Hamiltonian to go back to its original form, ψ_m an eigenstate of $H(t)$, and the integral is a closed line integral in the parameter space R (n-dimensional, where n is the number of parameters). Note that Berry obtained the equation above in 1984!

Let us now go back to Aharonov-Bohm. If we consider an infinitely long cylindrical solenoid, the field $\vec{B}$ is zero outside it, but the vector potential is not:

$$\vec{A}(r) = \frac{\mu_0 n I R^2}{2r}\hat{\phi} \quad \text{for } r > R$$

R = radius of solenoid
n = number of turns per unit of length
I = current in solenoid

$\vec{A}$ points in the same direction as the current I (i.e., the azimuthal direction) and indeed $\vec{\nabla} \times \vec{A} = \vec{B} = 0$ if $r > R$.

The classical Hamiltonian, in the presence of fields, is written in terms of the potentials:

$$H = \frac{1}{2m}(\underbrace{\vec{p} - q\vec{A}}_{\text{vector potential}})^2 + q\underbrace{V}_{\text{scalar potential}} \quad \text{where} \quad \vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The corresponding Schroedinger equation can be obtained by canonical substitution $\vec{p} \to -i\hbar\vec{\nabla}$:

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2 \psi(\vec{r},t) + qV\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}$$

Let us now note that we can greatly simplify it by using the following trick:

$$\psi(\vec{r},t) \equiv e^{ig(\vec{r})}\varphi(\vec{r},t) \quad \text{where} \quad g(\vec{r}) = \frac{q}{\hbar}\int_{\substack{\text{reference}\\\text{point}}}^{\vec{r}} \vec{A}(\vec{r}\,')\cdot d\vec{\ell}$$

$g(\vec{r})$ is the anti-derivative of $\vec{A}(\vec{r})$:

$$\vec{\nabla} g(\vec{r}) = \frac{q}{\hbar}\vec{A}(\vec{r})$$

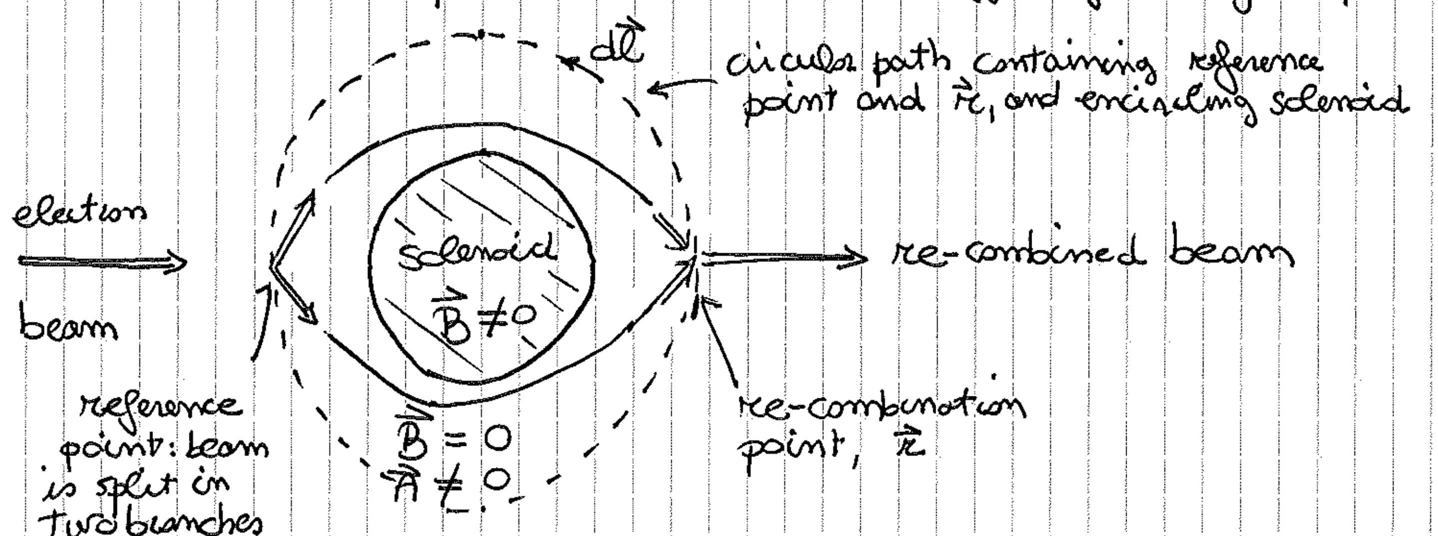
The integral $\int_{\substack{\text{reference}\\\text{point}}}^{\vec{r}}$ does not depend on the chosen path because $\vec{\nabla}\times\vec{A}=0$, rather, as long as

With this replacement, Schroedinger equation becomes:

$$-\frac{\hbar^2}{2m}\nabla^2\varphi(\vec{r},t) + qV\varphi(\vec{r},t) = i\hbar\frac{\partial\varphi(\vec{r},t)}{\partial t}$$

A free particle equation (in a potential $V(\vec{r})$)!

Let us sketch an experiment that shows the effect of the $ig(\vec{r})$ phase:



The beam is split in an upper and a lower paths. Let us take $d\vec{\ell}$ on a path from the reference point to $\vec{r}$, and vi ceversa (a closed loop).

We already noticed that, since $\vec{\nabla}\times\vec{A}=0$, it does not matter which path we choose (as long as we sketch it in areas where $\vec{\nabla}\times\vec{A}=0$). Let us take a circular path, to simplify calculations. We find:

$$|g_{upper}(\vec{r})| = |g_{lower}(\vec{r})| = \underbrace{\frac{1}{2}}_{\text{upper and lower paths are the same, modulo sign}}\frac{q}{\hbar}\underbrace{\oint}_{\text{closed loop}} \vec{A}\cdot d\vec{l} \underbrace{=}_{\text{stoke's theorem}} \frac{q}{2\hbar}\underbrace{\phi_B}_{\text{flux of }\vec{B}\text{ through loop: }\neq 0\text{, solenoid is inside loop}} \neq 0$$

We also note that $g_{upper}(\vec{r}) = -g_{lower}(\vec{r})$ because the electron trajectory is generally parallel to $\vec{A}$ in one of the paths, and anti-parallel to $\vec{A}$ in the other path. The wavefunction of the recombined beam will become:

$$\psi(\vec{r},t) = \frac{1}{2}\left(\psi_{upper}(\vec{r},t)+\psi_{lower}(\vec{r},t)\right) = \frac{1}{2}\left(e^{-ig(\vec{r})}+e^{ig(\vec{r})}\right)\underbrace{\varphi(\vec{r},t)}_{\text{free particle } (V(\vec{r})=0 \text{ in our experiment})}$$

where $g(\vec{r}) = |g_{upper}(\vec{r})| = |g_{lower}(\vec{r})|$

Then:

$$|\psi(\vec{r},t)|^2 = |\varphi(\vec{r},t)|^2\cdot\cos^2\left(\frac{q\,\phi_B}{2\hbar}\right)$$

the electrons will interfere when the beams are recombined, by an amount depending on magnetic flux in the solenoid, which the electrons <u>never</u> experience ($\vec{B}=0$ wherever electrons travel).

This effect cannot be explained by classical electromagnetism, and became a true quantum paradigm (many experiments in different contexts have been performed).

Applications: SQUID (superconducting quantum interference device)