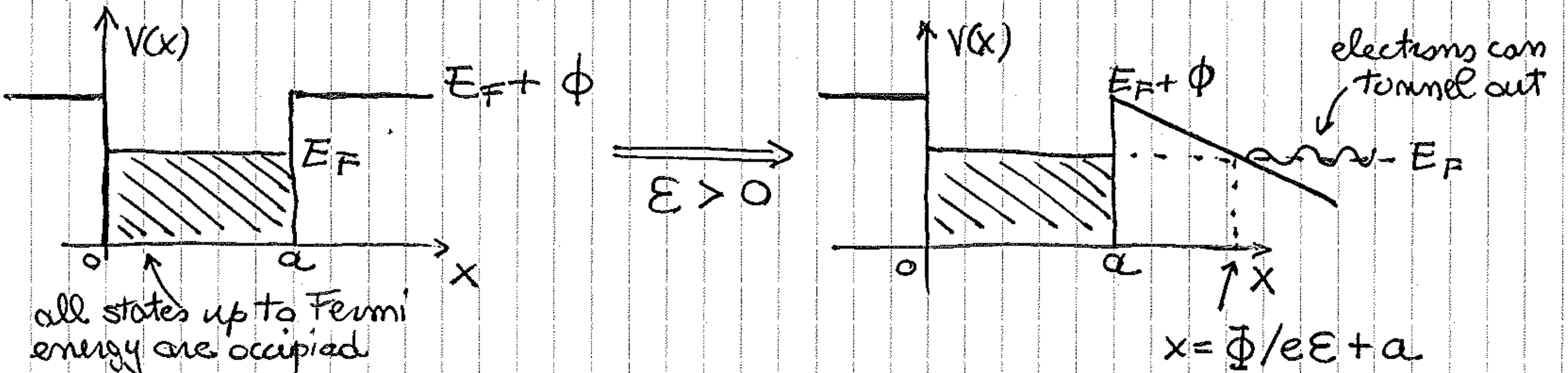
TUNNELLING EXAMPLES

1) COLD EMISSION BY FOWLER-NORDHEIM TUNNELLING

Let us take a piece of metal, make it into a sharp point, and hold it over a flat metallic surface, with a small gap. Let us do so in vacuum, and with an electric potential between tip and surface. We ask ourselves what is the current between tip and surface, as a function of the applied electric field. The current is due to electrons tunnelling out of metal (cold emission). Let us model a potential for the electrons. If we take a 1-D quantum well, with ∞ walls, we can not have any tunnelling. Let us start by making our well $E_F+\phi$ deep, where ϕ is the energy required to extract an electron from the metal and set it free (it is of the order of a few eV). Let us start with $\mathcal{E}$ = electric field = 0:



When $\mathcal{E} \neq 0$, the potential becomes $V(x) = E_F + \Phi - e\mathcal{E}(x-a)$. This is a job for WKB:

$$T = e^{-2\gamma} \quad \text{where} \quad \gamma = \frac{1}{\hbar}\int_a^{a+\Phi/e\mathcal{E}} \left|\sqrt{2m(E-V(x))}\right| dx \underset{x'=x-a}{=} \frac{1}{\hbar}\int_0^{\Phi/e\mathcal{E}} \left|\sqrt{2m(E_F-E_F-\phi+e\mathcal{E}x')}\right| dx' =$$

$$= \frac{1}{\hbar}\sqrt{2m}\int_0^{\phi/e\mathcal{E}} \sqrt{\phi - e\mathcal{E}x'}\, dx'$$

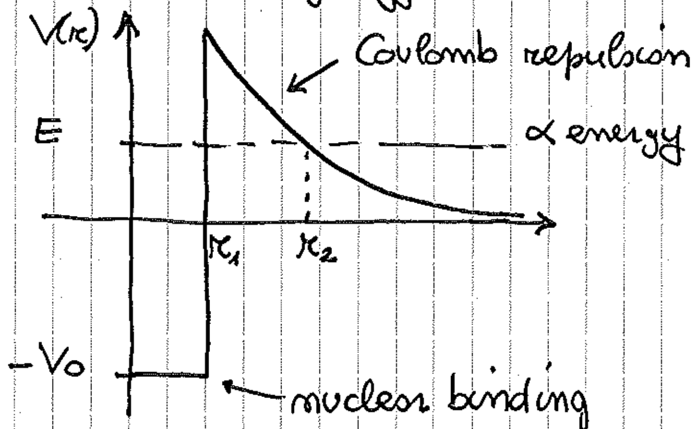
The integral can be solved easily, and results in:

$$T = e^{-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{\phi^{3/2}}{\mathcal{E}e}} \Longrightarrow \ln I = \text{log of tunnelling current} \propto \overset{\text{transmission coefficient}}{\ln T} \propto 1/\mathcal{E}$$

The log of the tunnelling current is proportional to the inverse of the applied electric field : this has been demonstrated experimentally. This is one of the first successes of quantum mechanics

2) GAMOW MODEL OF ALPHA DECAY

This is the first application of quantum mechanics to nuclear physics! Here is the idea : the alpha particle (a ^4He nucleus) is considered as a particle travelling in the rest of the nucleus of a radioactive element. It would naturally be repelled by the rest of the nucleus (they have the same-signed charge - positive), but first it has to pass a potential barrier that was known, in the case of uranium, to be twice as large than the energy of the emitted α particle. Gamow identified the escape mechanism as being a tunnelling effect. Here is the model :



$r_1 \simeq$ nuclear radius, order of 1 fm (10^{-15} m)

r_2 = turning point = $\dfrac{2Ze^2}{4\pi\varepsilon_0} \cdot \dfrac{1}{E}$

(2 : # protons in α particle
Z : # protons in rest of nucleus
E : energy of α particle - we can measure this in a lab)

Again, let us use WKB :

$$T = e^{-2\gamma} \quad \text{where} \quad \gamma = \frac{1}{\hbar}\int_{r_1}^{r_2} \sqrt{2m}\left(\frac{1}{4\pi\varepsilon_0}\,\frac{2Ze^2}{r} - E\right)^{1/2} dr = \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\left(\frac{r_2}{r}-1\right)^{1/2} dr$$

This integral is feasible, albeit horrible (check Griffiths example 8.2). But here is an interesting consequence. Imagine the α particle as travelling at speed v in the nucleus. It reaches the "wall" at r_1 every $2r_1/v$ seconds, and every time it has a probability $e^{-2\gamma}$ to escape. Therefore, the lifetime of the parent nucleus is $\tau = \frac{2r_1}{v} e^{+2\gamma}$; then $\underline{\log \tau \propto \sqrt{E}}$: this is experimentally confirmed!

EXTRA: a less quick recap of tunnelling

So far we assumed $E > V(x)$, and studied how $\psi(x)$ becomes when $V(x)$ is not a constant. But we started from a sinusoidal function, $\psi(x) = Ae^{ikx}$, and modified it into another, almost sinusoidal, function:

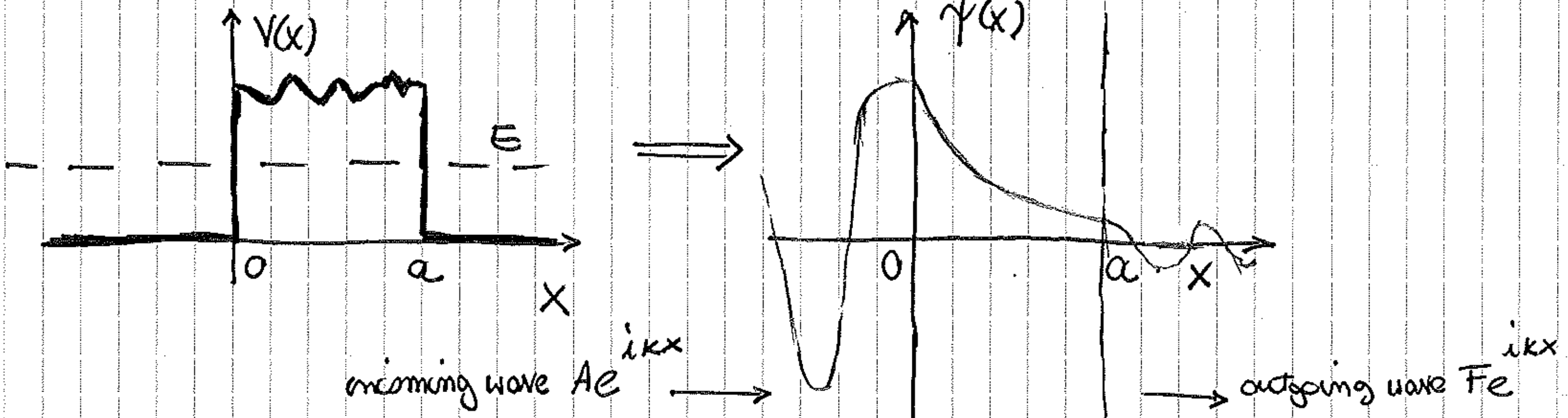
$$\psi_{WKB}(x) = \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int^{x} p(x')dx'} \quad \text{where } p(x) = \sqrt{2m\underbrace{(E-V(x))}_{E>V(x)}} = \text{real number}$$

If $E < V(x)$, we can save most of our calculations, and obtain a slightly different result:

$$\psi_{WKB}(x) = \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar}\int^{x} |p(x')| dx'}$$

where $|p(x)| = |\sqrt{2m(E-V(x))}| = |i\sqrt{2m(V(x)-E)}| = \sqrt{2m(V(x)-E)}$: almost the classical $p(x)$.

Let us picture a typical tunnelling problem:



$x < 0$ and $x > a$: $E > V(x) \Rightarrow$ solution is plain Ae^{ikx} wavefunction

$0 < x < a$: $E < V(x) \Rightarrow$ we use WKB here and obtain:

$$\psi_{WKB}(x) = \frac{C}{|p(x)|} \exp\left(-\frac{1}{\hbar}\int_{0}^{x} |p(x')| dx'\right)$$

Before the barrier, the solution is Ae^{ikx}, after the barrier it has the same functional form, but different amplitude: Fe^{ikx}. The ratio between the two amplitudes is given by: $\frac{|F|}{|A|} \propto e^{-\frac{1}{\hbar}\int_{0}^{a} |p(x)dx'}$

The part of $\psi_{WKB}(x)$ with a positive exponential does not play a role: it corresponds to an evanescent wave produced by a reflection on the right barrier (the one at $x = a$) of the exponentially decreasing $\psi_{WKB}(x)$ part with a negative exponential. If the wall/barrier is high and ~~be~~ wide, the coefficient of the $\psi_{WKB}(x)$ term with positive exponential is very small.

Back to tunnelling probability. The transmission coefficient, which corresponds to the tunnelling probability, is:

$$T \propto \frac{|F|^2}{|A|^2}$$

Hence:

$$T \simeq e^{-2\gamma} \quad \text{where} \quad \gamma = \frac{1}{\hbar}\int_0^a |p(x')\,dx'$$