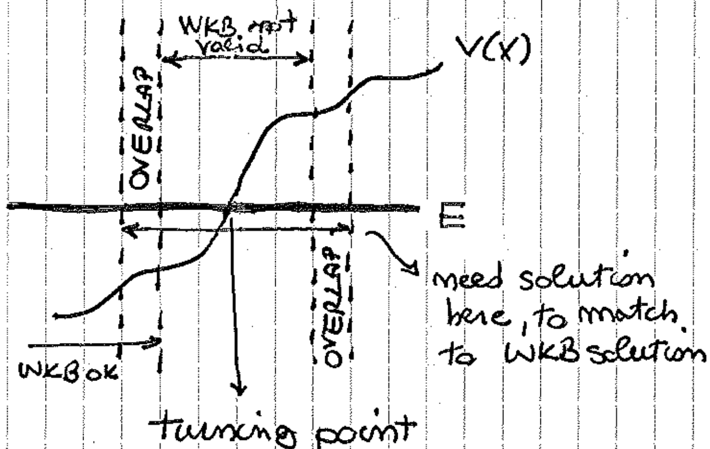
MORE ON WKB APPROXIMATION

We saw we have a problem at classical turning points. There, $p(x) \to 0$ and the WKB approximation breaks. This is not a problem when, in those points, $V(x) \to \infty$, thus forcing $\psi(x)$ to go to zero. But what if $V(x)$ is finite at a classical turning point? Here is a sketch of a possible case:



SOLUTION: we look for exact solution in range in which WKB not valid, and match it to the WKB solutions far enough to the turning point, where they are valid (note: if $V>E$, I get exponential decay)

We still need to do one approximation: let us approximate the potential at the turning point with a straight line:

$$V(x) \simeq E + \left.\frac{dV}{dx}\right|_{x=x_p} \cdot (x-x_p) \leftarrow \text{turning point}$$

Schroedinger equation becomes simple (at least, it does look so):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} + \left(E + \frac{dV}{dx}(x-x_p)\right)\psi_p = E\psi_p$$

where $\psi_p(x)$ is the solution valid at the turning point. For simplicity, let me also make the turning point be at 0. We obtain:

$$\frac{d^2\psi_p}{dx^2} = \alpha^3 x\,\psi_p \qquad \text{where } \alpha = \left(\frac{2m}{\hbar^2}\left.\frac{dV}{dx}\right|_{x=x_p}\right)^{1/3}$$

re-writing with change of variable $x \to z = \alpha x$, we obtain:

$$\frac{d^2\psi_p}{dz^2} = z\cdot\psi_p \implies \text{solutions are Airy functions}$$

$Ai(z)$ and $Bi(z)$

Let us focus on patching $\psi_{WKB}(x)$ and $\psi_p(x)$ on the overlap region where $E > V(x)$. Griffiths also does the $E < V(x)$ case, the right side of our sketch, but that is trivial. The left side of the sketch is more interesting.

Let us assume (they are built this way) the overlap regions to be far enough from x_p (the turning point) that WKB provides a valid solution, but close enough that we can approximate $V(x)$ with a straight line. Then:

$$p(x) \simeq \sqrt{2m\,(E - E - V'(0)\,x)} = \hbar\,\alpha^{3/2}\sqrt{x}$$

$V' = \frac{dV}{dx}$ again, let me put x_p at 0

In the left overlap region, we have:

$$\psi_{WKB}(x) = \frac{1}{\sqrt{p(x)}}\cdot\left[A e^{i\phi(x)} + B e^{-i\phi(x)}\right]$$ where

$$\hbar\,\phi(x) = \int_x^0 |p(x')|\,dx' \cong \hbar\,\alpha^{3/2}\int_x^0 \sqrt{x'}\,dx' = \frac{2}{3}\hbar\,(-\alpha x)^{3/2}$$

$$\psi_p(x) = C\,Ai(\alpha x) + D\,Bi(\alpha x)$$

in the limit $\alpha x << 0$ (large and negative):

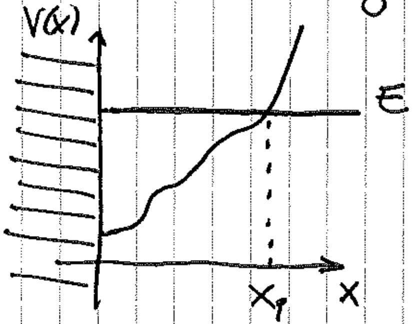
$$\psi_p(x) \simeq \frac{C}{\sqrt{\pi}\,(-\alpha x)^{1/4}}\sin\left(\frac{2}{3}(-\alpha x)^{3/2} + \frac{\pi}{4}\right)$$

Comparing the two expressions, we find the following CONNECTION FORMULAE for going from the left side to the right side of a classical transition point:

$$\psi(x) \simeq \begin{cases} \dfrac{2A}{\sqrt{p(x)}}\sin\left(\dfrac{1}{\hbar}\displaystyle\int_x^0 p(x')\,dx' + \dfrac{\pi}{4}\right); & x < 0 \\[2ex] \dfrac{A}{\sqrt{p(x)}}\,e^{-\frac{1}{\hbar}\int_x^0 p(x')\,dx'}; & x > 0 \end{cases}$$

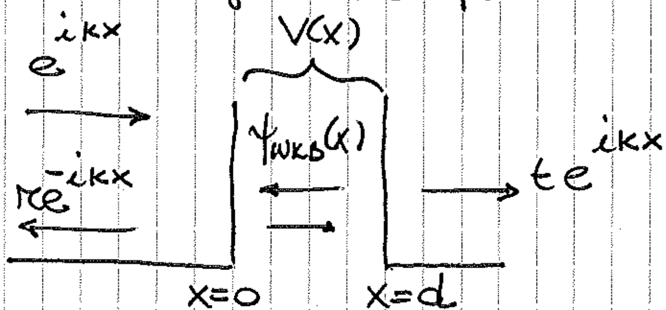
note the appearance of a phase $\pi/4$: that is the effect of having one soft wall (i.e., $V(x)$ finite). E.g., let us take a potential well in which one wall is not infinitely steep. Since $\psi(0)=0$, and the turning point is at some x_p value above 0, we get:

$$\psi(0)=0=\frac{1}{\hbar}\int_0^{x_p} p(x)\,dx+\frac{\pi}{4}=n\pi \implies \int_0^{x_p} p(x)\,dx=\left(n-\frac{1}{4}\right)\hbar\pi$$



the quantization condition has changed!

The WKB approximation is also very useful to calculate tunnelling rates through odd-shaped barriers. A very quick recap of tunnelling:



important note: now $E<V$, p becomes imaginary, and we get another factor i

if $E \ll V(x)$, then $\psi_{WKB}(x)=\frac{C_+}{\sqrt{|p(x)|}}e^{|\phi(x)|}+\frac{C_-}{\sqrt{|p(x)|}}e^{-|\phi(x)|}$

we have an exponential decay inside the barrier. The C_+ term describes a wave reflected from the wall at $x=d$, caused by the incident wave C_-: it has to be small $\Rightarrow$ we can assume $C_+\simeq 0$. We finally obtain:

$$T=\text{transmission rate} \propto \frac{|\psi_{WKB}(d)|^2}{|\psi_{WKB}(0)|^2}=e^{-2|\phi(d)|}$$

where $\phi(d)=\frac{1}{\hbar}\int_0^d |p(x)|\,dx$

now $p(x)$ is purely imaginary: $p(x)^2=2m(E-V(x))<0$!

I need to use absolute value