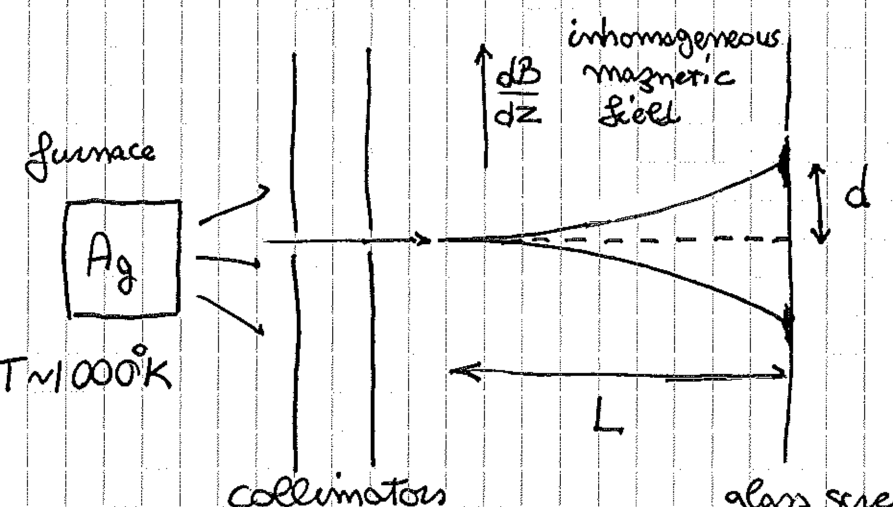
STERN-GERLACH EXPERIMENT

We saw that using the idea of starting from the relativistic energy we need to move to a 4-dimensional representation of electrons and positrons (anti-matter appears natural consequence of math). We have a total of 4 eigenvalues; what degree of freedom do these two wavefunction components correspond to (two components for e^- and two for e^+)?

Stern and Gerlach set out an experiment that demonstrated that atomic-scale system have quantized properties. In 1922, they built the apparatus sketched below:



Back-of-the-envelope estimation of effect for an atom:

$$E = -\vec{\mu}\cdot\vec{B}$$ ($\vec{\mu}$ is magnetic moment of atom, order of magnitude: Bohr magneton $5.8\cdot 10^{-5}\ eV/T$)

$$\Downarrow$$

$$F = -\frac{dE}{dz} = \vec{\mu}\cdot\frac{d\vec{B}}{dz} = \langle \mu_z \rangle \frac{dB}{dz}$$

atom accelerates due to this force, and ultimately lands at distance d:

$$d = \frac{1}{2}a\cdot t^2 = \frac{1}{2}\frac{F}{m}\cdot\left(\frac{L}{v}\right)^2$$

where v can be estimated from temperature of furnace: $\frac{1}{2}mv^2 = \frac{3}{2}k_B\cdot T$ (Boltzmann)

Let us put in all numbers:

$$d \approx \frac{1}{2} \langle \mu_z \rangle \frac{dB}{dz} \cdot L^2 \cdot \frac{1}{3 k_B T} = \frac{1}{6} \cdot \frac{5.8 \cdot 10^{-5}\, eV/T \cdot \overbrace{\frac{10T}{cm}}^{B \text{ gradient}} \cdot (3.5\, cm)^2}{0.1\, eV} =$$

$$= 10^{-2}\, cm = 100\, \mu m$$

Interesting point is that Ag atom has total orbital angular momentum equal to 0!

Observed splitting therefore due to a magnetic moment associated with <u>intrinsic</u> angular momentum, without a classical mechanics equivalent: it is the spin of the (outermost) electron:

$$\langle \mu_z \rangle \propto -\mu_B \frac{\langle S_z \rangle}{\hbar} \; ; \; \langle S_z \rangle = \pm \frac{\hbar}{2}$$

These two states (spin up and down) correspond to the two electron components of Dirac wavefunctions.

MORE SPIN + MAGNETIC FIELDS ← skipped: done in discussion session

The spin couples to magnetic fields via its associated magnetic moment:

$$\vec{\mu} = -g \frac{\mu_B}{\hbar} \vec{S}$$

The Hamiltonian term is:

$$H = -\vec{\mu} \cdot \vec{B} = g \frac{\mu_B}{\hbar} \vec{S} \cdot \vec{B}$$

not really 2: QED corrections

measured to $< 10^{-12}$

where g is the Thomas g-factor; its value is 2, and we need it to account for the relativistic Lorentz boost

Let us take a vertical, constant B field:

$$\vec{B} = B_z \hat{z}$$

Using Pauli's matrices, the Hamiltonian is easy:

$$H = g \frac{\mu_B}{\hbar} \vec{S} \cdot \vec{B} = 2 \frac{\mu_B}{\hbar} \cdot \frac{\hbar}{2} \sigma_z \cdot B_z = \mu_B \sigma_z B_z = \begin{pmatrix} \mu_B B_z & 0 \\ 0 & -\mu_B B_z \end{pmatrix}$$

eigenvalues and corresponding eigenstates: $\quad e^{-iEt/\hbar}$

$$E = \mu_B B_z \qquad |\uparrow(t)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu_B B_z t/\hbar} \qquad \text{up state}$$

$$E = -\mu_B B_z \qquad |\downarrow(t)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mu_B B_z t/\hbar} \qquad \text{down state}$$

Note: up state has spin anti-parallel to $\vec{B}$: $\quad \uparrow B_z \ \downarrow S_z$
viceversa in the case of down state

What if we had started from an eigenvector of S_x?

If I diagonalize $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, I obtain the following two

eigenstates: $\quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, with eigenvalue $\hbar/2$

$\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, with eigenvalue $-\hbar/2$

What happens if I prepare the state $|+\rangle$ and apply $\vec{B} = B_z \hat{z}$?

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow |+(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\mu_B B_z t/\hbar} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\mu_B B_z t/\hbar}$$

What happens to the expectation value $\langle S_x \rangle$? We started in an eigenstate of S_x, one with $\langle S_x(t=0)\rangle = \hbar/2$

Now:

this means I need to take complex conjugate!

$$\langle S_x(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\mu_B B_z t/\hbar} & e^{i\mu_B B_z t/\hbar}\end{pmatrix}^* \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\mu_B B_z t/\hbar} \\ e^{i\mu_B B_z t/\hbar}\end{pmatrix}$$

$\langle +(t) | S_x | +(t) \rangle$

row vector

column vector

$$= \frac{\hbar}{4}\begin{pmatrix} e^{i\mu_B B_z t/\hbar} & e^{-i\mu_B B_z t/\hbar}\end{pmatrix}\begin{pmatrix} e^{+i\mu_B B_z t/\hbar} \\ e^{-i\mu_B B_z t/\hbar}\end{pmatrix} =$$

I took complex conjugate

multiplication with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ swaps position

$$= \frac{\hbar}{4}\left(e^{2i\mu_B B_z t/\hbar} + e^{-2i\mu_B B_z t/\hbar}\right) =$$

$$= \frac{\hbar}{2}\cdot \cos\left(2\mu_B B_z t/\hbar\right)$$ oscillations with $\omega = \frac{\Delta E}{\hbar}$

ΔE = energy difference between the two eigenstates

Similarly: $\langle + | S_y | + \rangle = \frac{\hbar}{2}\sin\left(2\mu_B B_z t/\hbar\right)$ (out of phase by $\pi/2$)

The <u>fictitious</u> vector $\langle S_x \rangle \hat{x} + \langle S_y \rangle \hat{y}$ rotates around $\vec{B}$ as a classical angular momentum performs a precession around a force causing a torque