# PHOTONS IN A BOX

The energy density of photons in a black-body radiation source is the Planck black-body distribution function:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \cdot \frac{1}{e^{\hbar\omega/k_BT}-1}$$

We can now derive it using our knowledge of quantum statistical mechanics.

Here are three crucial properties of photons:

1) they are <u>massless</u> spin-1 bosons: they exist only in <u>2</u> polarization states, which carry angular momentum $+\hbar$ (left-circularly polarized) or $-\hbar$ (right-circularly polarized)
2) they carry energy $E=\hbar\omega$, as described in Einstein's explanation of photoelectric effect
3) they carry momentum $p = \hbar K = h/\lambda = E/c$, as obtained from Compton scattering experiments

We can use Bose-Einstein distribution, with an additional note: photons are continuously absorbed and re-emitted by the walls of our box. Therefore, their number $N$ is <u>not</u> constant. Releasing this constraint is equivalent to setting the chemical potential $\mu = 0$. Hence we obtain:

$$n_s = \frac{g_s}{e^{E_s/k_BT}-1}$$ : number of identical photons with energy $E_s$

We need to find the energy levels $E_s$ and their degeneracy $g_s$.

Let us consider an empty cubic box with perfectly conducing walls, so that the tangential component of $\vec{E}$ is 0 on the walls. The box has volume $V = a \cdot a \cdot a$. Let us solve Maxwell's equation for an electric field in free space:

$$\nabla^2 \vec{E} + K^2 \vec{E} = 0$$

The solution is the following: each component of $\vec{E}$ can be written as a product of three functions, $X(x)Y(y)Z(z)$, and the equation separates. E.g.:

$$E_x(x,y,z) = X(x)Y(y)Z(z)$$

$$\nabla^2 E_x + K^2 E_x = 0 \implies \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\text{depends on } x \text{ only}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\text{depends on } y \text{ only}} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\text{depends on } z \text{ only}} = -K^2$$

$$\implies \frac{X''}{X} = -K_x^2 \; ; \; \frac{Y''}{Y} = -K_y^2 \; ; \; \frac{Z''}{Z} = -K_z^2 \quad \text{and} \quad K^2 = K_x^2 + K_y^2 + K_z^2$$

Boundary conditions require wavenumber $K$ to be quantized:

$$K_x = \frac{l\pi}{a} \qquad K_y = \frac{m\pi}{a} \qquad K_z = \frac{n\pi}{a} \qquad l, m, n \text{ positive integers}$$

We now find the energy levels $E_s$:

$$E_{lmn} = P_{lmn} \cdot c = \hbar c K = \hbar c \frac{\pi}{a}\sqrt{l^2+m^2+n^2}$$

(the index $s$ is replaced by quantum numbers $l, m, n$)

What is the degeneracy of the energy level $E_{lmn}$, $g_{lmn}$? Let us use the same trick used when we studied the Fermi gas of free electrons.

We go to the K-space, where each state occupies a volume proportional to $1/V$. When $l, m$ and $n$ are large, the states become so dense that I can treat them as if they were continuous.

All states with same energy will lie on a spherical octant centered at the origin in K-space. A shell of thickness $dk$, at radius $k$, will contain:

$$\#\text{states} = \frac{\text{volume of shell}}{\text{volume per state}} = \underbrace{\frac{1}{8}}_{\text{octant}} \underbrace{4\pi k^2 dk}_{\text{volume of spherical shell}} \cdot \underbrace{\frac{1}{\pi^3/V}}_{\text{volume in K-space of a } \Delta K_x \Delta K_y \Delta K_z \text{ cell}} \cdot \underbrace{2}_{\text{two possible polarization states}}$$

Let us express the degeneracy using $\omega = kc$ :

$$g(k)\,dk = \frac{k^2 V}{\pi^2}\,dk = g(\omega)\,d\omega \quad \text{where } g(\omega) = \frac{V\omega^2}{\pi^2 c^3}$$

Finally:

$$n(\omega) = \frac{g(\omega)}{e^{\hbar\omega/k_BT}-1} = \frac{V\omega^2}{\pi^2 c^3}\cdot\frac{1}{e^{\hbar\omega/k_BT}-1}$$

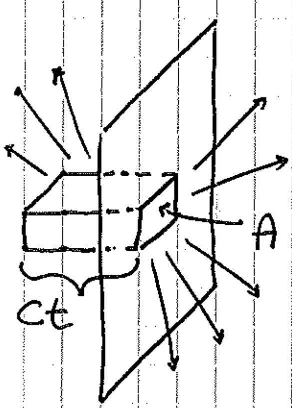
$n(\omega)$ → number of photons with energy $\hbar\omega$

The <u>energy density</u> in the box is equal to $n(\omega)$ divided by the volume of the box ($V$) times the energy of the occupied state ($\hbar\omega$):

$$\boxed{\rho(\omega)} = n(\omega)\cdot\hbar\omega/V = \boxed{\frac{\hbar\omega^3}{\pi^2 c^3}\;\frac{1}{e^{\hbar\omega/k_BT}-1}}$$

PLANCK'S BLACK BODY RADIATION FORMULA

Above we found the energy per unit of volume and frequency. What about the power (energy per time) irradiated per unit of area and steradian? Let us imagine cutting a small hole of area $A$ in our box.




In a time $t$, the photons contained in a volume $A\cdot c\cdot t$ will irradiate in all directions; half of them will go outside the box, covering a $2\pi$ solid angle.

The energy contained in the volume under consideration is $\rho(\omega)\,A\cdot c\cdot t$, therefore:

$$\frac{\text{power}}{\text{area}\cdot\text{steradian}\cdot\text{frequency}} = \frac{\rho(\omega)Act}{A\cdot t\cdot 2\pi\cdot 2} = \frac{c}{4\pi}\rho(\omega) = \frac{\hbar\omega^3}{4\pi^3 c^2}\;\frac{1}{e^{\hbar\omega/k_BT}-1}$$

To get the total irradiated power, I need to integrate over frequencies and solid angle:

$$\boxed{\frac{P(\text{total})}{A} = \text{total power per unit of area}} = \int_0^\infty d\omega\,\frac{\hbar\omega^3}{4\pi^3c^2}\,\frac{1}{e^{\hbar\omega/k_BT}-1} = \frac{\hbar}{4\pi^3c^2}\left(\frac{k_BT}{\hbar}\right)^4\underbrace{\int_0^\infty\frac{x^3}{e^x-1}\,dx}_{\pi^4/15} = \boxed{\sigma T^4}$$

STEFAN-BOLTZMANN LAW