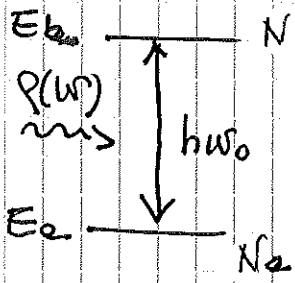
SPONTANEOUS EMISSION

Let us study a two-level system, with energy levels E_a and E_b having, respectively, N_a and N_b particles in each. We saw, in time-dependent perturbation theory, that we can stimulate transitions by applying, for example, an electromagnetic field. Another way to see the problem is the following: let us imagine to have $N = N_a + N_b$ identical atoms, N_b of which are in a state with energy E_b, and the remaining N_a are in a state with energy E_a. The EM field will stimulate transitions from a to b, and from b to a.
↳ excitation ↳ stimulated emission

Here is the situation:



$\hbar\omega_0 = E_b - E_a$

Let us write the rate equation:

$$\frac{dN_b}{dt} = \underbrace{N_a \cdot B \cdot \rho(\omega_0)}_{\text{(absorption)}} - \underbrace{N_b \cdot B \cdot \rho(\omega_0)}_{\text{(stimulated emission)}} - \underbrace{N_b \cdot A}_{\text{(spontaneous emission)}}$$

where $B = \frac{\pi |p|^2}{3\varepsilon_0 \hbar^2}$, such that $B \cdot \rho(\omega_0) = R_{a\to b} = R_{b\to a}$. Therefore:

1) $N_a \cdot B \cdot \rho(\omega_0) = N_a \cdot R_{a\to b}$: number of particles/atoms in state a that transition to state b per unit of time : radiation $\rho(\omega)$ is ABSORBED
2) $N_b\, B\, \rho(\omega_0) = N_b \cdot R_{b\to a}$: number of particles/atoms in state b that drop to state a per unit of time : STIMULATED EMISSION by radiation $\rho(\omega)$
3) $N_b \cdot A =$ number of particles/atoms in state b that SPONTANEOUSLY drop to state a

What is A? Let us look at our system when it reaches a steady state: $dN_b/dt = 0$

Then:

$$N_a B \rho(\omega_0) - N_b B \rho(\omega_0) - N_b A = 0$$

$$\rho = \frac{A}{\frac{N_a}{N_b} B - B}$$

Let us now suppose that our atoms are in thermal equilibrium with the ambient photons (i.e., the particles we associate with EM fields). Statistical mechanics says that:

$$N_a \propto e^{-E_a/k_B T} \qquad \text{where } k_B = \text{Boltzmann constant} = \frac{1/40\ eV}{300\ K}$$

Then:

$$N_a/N_b = e^{\hbar\omega_0/k_B T}$$

And:

$$\rho(\omega_0) = \frac{A}{B} \cdot \frac{1}{e^{\hbar\omega_0/k_B T} - 1}$$

Let us compare this expression with Planck's blackbody formula for the energy density of thermal radiation:

$$\rho_{\text{Blackbody}}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \cdot \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

Comparing the two, we find that $A/B = \hbar\omega_0^3/\pi^2 c^3$. Time-dependent perturbation theory indicated that $B = \pi |p|^2 / 3\varepsilon_0 \hbar^2$, finally:

$$\boxed{A = \frac{\omega_0^3 |p|^2}{3\pi\varepsilon_0 \hbar c^3}}$$

SPONTANEOUS EMISSION RATE

A couple of notes to meditate upon:

- we used the same B both for absorption and stimulated emission because we know that is the case from perturbation theory. When Einstein worked on this, he had to "invent" stimulated emission, and was forced to make it equally probable as absorption, in order to reproduce Planck's result
- spontaneous emission should trigger some complaint: if there is no perturbation, a particle in state b just stays there: it is in an eigenstate of H_0, and will just stay there forever. The final explanation of what happens is not in the quantum mechanics we did so far.

Spontaneous emission rate is defined by the Einstein A coefficient. Let us imagine we have a large number of atoms in state b. As a result of spontaneous emission, their number will decrease:

$$\frac{dN_b}{dt} = -N_b A \quad \Longrightarrow \quad N_b(t) = N_b(0)\, e^{-\frac{t}{\tau}} \quad \text{where } \tau = \frac{1}{A} \text{ is the}$$

LIFETIME of state b

Let us note that we are studying dipole transitions, in which A is defined as soon as we choose the initial state (b) and the final state (a):

$$A = \frac{\omega_0^3 |p|^2}{3\pi \varepsilon_0 \hbar c^3}$$

where $\omega_0 = (E_b - E_a)/\hbar$ and $\vec{p} = \langle a | e\vec{r} | b \rangle$

(remember that we obtained the factor $1/3$ averaging $|\vec{p}\cdot\hat{n}|^2$ over all possible $\hat{n}$ directions: unpolarized $\vec{E}$)

Let us also note that if $|b\rangle$ can decay to more states, $|a_1\rangle, |a_2\rangle \ldots$, then the transition rates add:

$$A = A_1 + A_2 + A_3 + \ldots \quad \Longrightarrow \quad \tau = \frac{1}{A_1 + A_2 + A_3 + \ldots}$$

Let us estimate the spontaneous emission rate, e.g., from excited hydrogen atom states:

$$\langle \vec{r} \rangle \sim a_0 = \frac{\hbar}{mc\alpha}$$

$\langle \vec{r} \rangle$: dipole part; a_0: Bohr radius; α: fine structure constant, $\frac{e^2}{\hbar c}$

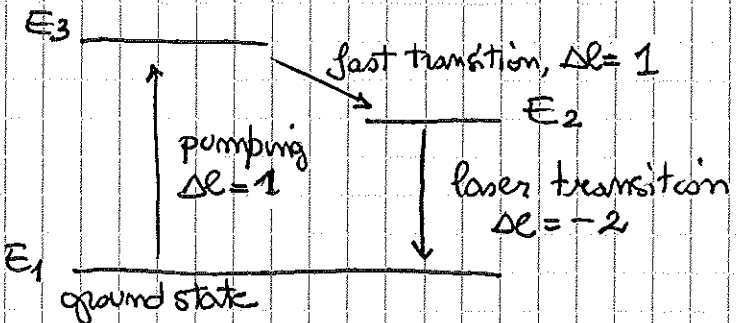
$$\omega_0 \sim \frac{E_0}{\hbar} = \frac{1}{2}\frac{mc^2\alpha^2}{\hbar}$$

$$\Rightarrow A \sim \left(\frac{mc^2\alpha^2}{\hbar}\right)^3 \cdot \left(\frac{\hbar e}{mc\alpha}\right)^2 \cdot \frac{1}{\hbar c^3} \sim \frac{mc\alpha^4 e^2}{\hbar^2} \sim \omega_0 \alpha^3 \sim 10^{-6}\,\omega_0$$

(ω_0^3; $|p|^2$)

spontaneous emission rate is <u>much</u> slower than internal dynamics: perturbation theory is valid.

BONUS (if time allows): LASER



the idea is that I can pump particles in the state E_3, from which they decay quickly to E_2 (both pumping and $E_3 \to E_2$ transitions are $\Delta\ell = 1$ and can happen via dipole). However, the $E_2 \to E_1$ transition is not allowed in dipole approximation: $\Delta\ell = -2$, I need a quadrupole. Hence, it is very slow, and I can obtain a <u>population inversion</u> by getting more atoms in state E_2 than in state E_1. At this point, a spontaneous emission from $E_2 \to E_1$ will (if I keep the photon in the laser cavity) stimulate the emission of more and more identical photons (same wavelength, same phase): this is my laser: coherence is lost when another photon is spontaneously emitted (with a different phase) and I start getting overlapping copies of this new photon