

## OSCILLATING MAGNETIC FIELD: EXACT SOLUTION

Let us take a spin-1/2 particle, at rest, in a static magnetic field  $\vec{B} = B_z \hat{z}$ . The Hamiltonian will be:

$$H^0 = -\vec{\mu} \cdot \vec{B} \quad \text{where} \quad \vec{\mu} = g \mu_B \vec{S} / \hbar \quad (g = -2 \text{ for electrons})$$

We will show that if we add, as a perturbation, a circularly-polarized oscillating magnetic field, we can find an exact solution.

Let our perturbing magnetic field be:

$$\vec{B}'(t) = B' (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y})$$

Using Pauli's matrices, and the spinor notation  $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ :

$$H' = \mu_B \vec{B}' \cdot \frac{\vec{\sigma}}{\hbar} = \mu_B B' \begin{pmatrix} 0 & \cos(\omega t) - i \sin(\omega t) \\ \cos(\omega t) + i \sin(\omega t) & 0 \end{pmatrix} =$$

$$= \mu_B B' \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$$

Let us indicate with  $|\chi(t)\rangle = C_\uparrow(t) |\uparrow(t)\rangle + C_\downarrow(t) |\downarrow(t)\rangle$  our solution of the following equation:

$$(H^0 + H') |\chi(t)\rangle = i\hbar \frac{\partial |\chi(t)\rangle}{\partial t}$$

Please note that:

$$|\uparrow(t)\rangle = e^{-iB_z \mu_B t / \hbar} |\uparrow\rangle$$

$$|\downarrow(t)\rangle = e^{+iB_z \mu_B t / \hbar} |\downarrow\rangle$$

$|\uparrow(t)\rangle$  and  $|\downarrow(t)\rangle$  are the solution of the equation:

$$H^0 |\chi(t)\rangle = i\hbar \frac{\partial |\chi(t)\rangle}{\partial t}$$

where  $H^0 = -\vec{\mu} \cdot \vec{B} = \mu_B B_z \sigma_z = \begin{pmatrix} \mu_B B_z & 0 \\ 0 & -\mu_B B_z \end{pmatrix}$

We saw we can write the following equations for  $C_{\uparrow}(t)$  and  $C_{\downarrow}(t)$ :

$$\begin{cases} \dot{C}_{\uparrow}(t) = -\frac{i}{\hbar} \langle \uparrow | H' | \downarrow \rangle e^{i\omega_0 t} C_{\downarrow}(t) \\ \dot{C}_{\downarrow}(t) = -\frac{i}{\hbar} \langle \downarrow | H' | \uparrow \rangle e^{-i\omega_0 t} C_{\uparrow}(t) \end{cases}, \text{ where } \omega_0 = \frac{E_{\uparrow} - E_{\downarrow}}{\hbar} = \frac{2B_z \mu_B}{\hbar}$$

Those two matrix elements are easy:

$$\langle \uparrow | H' | \downarrow \rangle = \mu_B B' (1 \ 0) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mu_B (0 \ e^{-i\omega t}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mu_B B' e^{-i\omega t}$$

$$\langle \downarrow | H' | \uparrow \rangle = \mu_B B' (0 \ 1) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_B B' (e^{i\omega t} \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_B B' e^{i\omega t}$$

Then, I can write:

$$\begin{cases} \textcircled{A} \dot{C}_{\uparrow}(t) = -i \frac{\mu_B B'}{\hbar} e^{i(\omega_0 - \omega)t} C_{\downarrow}(t) \\ \textcircled{B} \dot{C}_{\downarrow}(t) = -i \frac{\mu_B B'}{\hbar} e^{-i(\omega_0 - \omega)t} C_{\uparrow}(t) \end{cases}$$

This is a system of first-order differential equations, which I can transform into a single second-order ordinary differential equation. Let me make the replacement  $V = 2\mu_B B'$  ( $V$  can be thought of as being the energy gap between levels due to the perturbation only).

Let us take the time derivative of  $\textcircled{B}$ :  $\textcircled{A}$

$$\textcircled{C} \ddot{C}_{\uparrow} = i(\omega_0 - \omega) \dot{C}_{\uparrow} - i \frac{V}{2\hbar} e^{i(\omega_0 - \omega)t} \dot{C}_{\downarrow}(t)$$

derivative of exponential
I can write this as  $\dot{C}_{\uparrow} / C_{\downarrow}$

Take  $\textcircled{A}$  and multiply it by  $\textcircled{B}$ , side by side

$$\dot{C}_{\uparrow} \dot{C}_{\downarrow} = -\frac{V^2}{4\hbar^2} C_{\downarrow} C_{\uparrow}$$

We generally obtain (note the sign flip:  $i(\omega - \omega_0) = -i(\omega_0 - \omega)$ ):

$$\ddot{C}_\uparrow = -i(\omega - \omega_0) \dot{C}_\uparrow - \frac{V^2}{4\hbar^2} C_\uparrow$$

This is the same equation as a LCR circuit, a mass on a spring with friction: it is a damped oscillator. The solution has the form  $e^{\lambda t}$ , where  $\lambda$  are the solutions of the algebraic equation:

$$\lambda^2 + i(\omega - \omega_0)\lambda + V^2/4\hbar^2 = 0$$

$$\text{Solution: } \lambda = \frac{-i(\omega - \omega_0) \pm \sqrt{-(\omega_0 - \omega)^2 - V^2/\hbar^2}}{2} = i\left(-\frac{\omega - \omega_0}{2} \pm \omega_R\right)$$

$$\omega_R = \frac{1}{2} \sqrt{(\omega - \omega_0)^2 + V^2/\hbar^2}$$

RABI FLOPPING FREQUENCY

(if  $V$  is complex, use  $|V|^2$  here)

Let us massage a bit our solution:

$$C_\uparrow(t) = e^{\lambda t} = e^{-i\frac{\omega - \omega_0}{2}t} (a e^{i\omega_R t} + b e^{-i\omega_R t})$$

$a$  and  $b$  are fixed by initial conditions. Let us say that we were, at  $t=0$ , in  $|\downarrow\rangle$ . Then:  $C_\uparrow(0) = 0$  and  $C_\downarrow(0) = 1 \Rightarrow$

$$C_\uparrow(0) = 0 \Rightarrow a = -b$$

$$C_\downarrow(0) = 1 = i \frac{2\hbar}{V} e^{i\omega_0 t} \cdot \dot{C}_\uparrow(0) = i \frac{2\hbar}{V} \cdot 1 \cdot \left(-i\left(\frac{\omega - \omega_0}{2} - \omega_R\right) \cdot a - i\left(\frac{\omega - \omega_0}{2} + \omega_R\right) b\right)$$

use equation (A)

take derivative before setting  $t=0$ !

$$= i \frac{2\hbar}{V} \cdot (-i) (-2\omega_R) a \Rightarrow a = -b = \frac{-V}{4\hbar\omega_R}$$

$$b = -a$$

$$\text{One last operation: } a e^{i\omega_R t} + b e^{-i\omega_R t} \stackrel{b=-a}{=} a (e^{i\omega_R t} - e^{-i\omega_R t}) = 2ia \sin(\omega_R t).$$

We can finally write the full, exact solution:

$$C_{\downarrow}(t) = e^{i(\omega - \omega_0)t/2} \left( \cos(\omega_{\pm} t) + i \frac{\omega_0 - \omega}{2\omega_{\pm}} \sin(\omega_{\pm} t) \right)$$

$$C_{\uparrow}(t) = -\frac{iV}{2\hbar\omega_{\pm}} e^{i(\omega_0 - \omega)t/2} \sin(\omega_{\pm} t)$$

Let us test if it makes sense:

1) - at  $t=0$ :  $C_{\downarrow}(0) = 1$ ,  $C_{\uparrow}(0) = 0$ : great

2) - for weak driving conditions,  $V \ll \hbar(\omega - \omega_0)$ , it should match our approximation. Let us check this limit:

$$\omega_{\pm} = \frac{1}{2} \sqrt{(\omega - \omega_0)^2 + V^2/\hbar^2} \approx \frac{\omega - \omega_0}{2}$$

$$C_{\uparrow}(t) \approx -\frac{iV}{\hbar(\omega - \omega_0)} e^{i(\omega_0 - \omega)t/2} \sin\left(\frac{\omega - \omega_0}{2} t\right)$$

$$|C_{\uparrow}(t)|^2 \approx \frac{V^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega - \omega_0}{2} t\right)}{(\omega - \omega_0)^2}$$

: exactly our first-order solution (almost: factor of 2<sup>2</sup> difference: here we have both B<sub>x</sub> and B<sub>y</sub>, before just B<sub>x</sub>!)

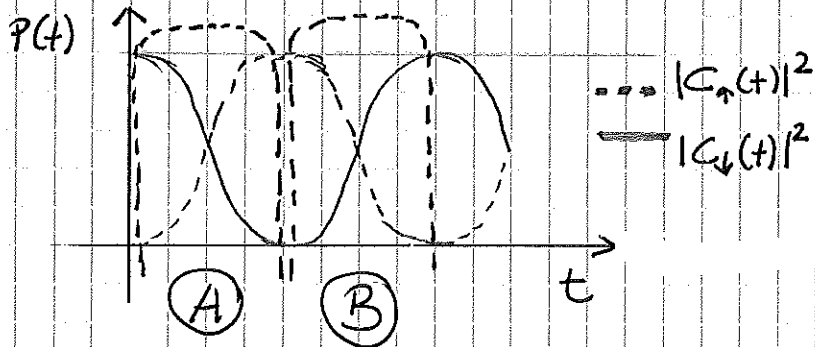
3) - we now have a solution also for  $\omega = \omega_0$ , where our first-order solution fails ( $P(\omega) \propto t^2$ ):

$$\omega_{\pm} = V/2\hbar$$

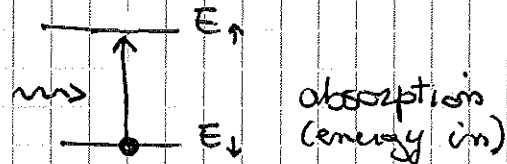
$$|C_{\uparrow}(t)|^2 = \sin^2(\omega_{\pm} t)$$

$$|C_{\downarrow}(t)|^2 = \cos^2(\omega_{\pm} t)$$

now probabilities are bounded by 1



(A)



(B)

