OSCILLATING MAGNETIC FIELD: EXACT SOLUTION

Let us take a spin-1/2 particle, at rest, in a static magnetic field $\vec{B} = B_Z \hat{z}$. The Hamiltonian will be:

$$H^0 = -\vec{\mu} \cdot \vec{B} \quad \text{where} \quad \vec{\mu} = g \mu_B \vec{S}/\hbar \qquad (g = -2 \text{ for electrons})$$

We will show that if we add, as a perturbation, a circularly-polarized oscillating magnetic field, we can find an exact solution.
Let our perturbating magnetic field be:

$$\vec{B}'(t) = B'(\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y})$$

Using Pauli's matrices, and the spinor notation $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$H' = \mu_B \vec{B}' \cdot \vec{\sigma} = \mu_B B' \begin{pmatrix} 0 & \cos(\omega t) - i\sin(\omega t) \\ \cos(\omega t) + i\sin(\omega t) & 0 \end{pmatrix} =$$

$$= \mu_B B' \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$$

Let us indicate with $|\chi(t)\rangle = C_\uparrow(t)|\uparrow(t)\rangle + C_\downarrow(t)|\downarrow(t)\rangle$ our solution of the following equation:

$$(H^0 + H')|\chi(t)\rangle = i\hbar \frac{\partial|\chi(t)\rangle}{\partial t}$$

Please note that: $|\uparrow(t)\rangle = e^{-iB_Z\mu_B t/\hbar}|\uparrow\rangle$

$|\downarrow(t)\rangle = e^{+iB_Z\mu_B t/\hbar}|\downarrow\rangle$

$|\uparrow(t)\rangle$ and $|\downarrow(t)\rangle$ are the solution of the equation:

$$H^0|\chi(t)\rangle = i\hbar \frac{\partial|\chi(t)\rangle}{\partial t}$$

where $H^0 = -\vec{\mu}\cdot\vec{B} = \mu_B B_Z \sigma_z = \begin{pmatrix} \mu_B B_Z & 0 \\ 0 & -\mu_B B_Z \end{pmatrix}$

We saw we can write the following equations for $C_\uparrow(t)$ and $C_\downarrow(t)$:

$$\begin{cases} \dot{C}_\uparrow(t) = -\frac{i}{\hbar} \langle \uparrow | H' | \downarrow \rangle e^{i\omega_0 t} C_\downarrow(t) \\ \dot{C}_\downarrow(t) = -\frac{i}{\hbar} \langle \downarrow | H' | \uparrow \rangle e^{-i\omega_0 t} C_\uparrow(t) \end{cases}, \text{ where } \omega_0 = \frac{E_\uparrow - E_\downarrow}{\hbar} = \frac{2 B_z \mu_B}{\hbar}$$

Those two matrix elements are easy:

$$\langle \uparrow | H' | \downarrow \rangle = \mu_B B' (1 \; 0) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mu_B B' (0 \; e^{-i\omega t}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mu_B B' e^{-i\omega t}$$

$$\langle \downarrow | H' | \uparrow \rangle = \mu_B B' (0 \; 1) \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_B B' (e^{i\omega t} \; 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_B B' e^{i\omega t}$$

Then, I can write:

$$\begin{cases} Ⓐ \; \dot{C}_\uparrow(t) = -i \frac{\mu_B B'}{\hbar} e^{i(\omega_0 - \omega)t} C_\downarrow(t) \\ Ⓑ \; \dot{C}_\downarrow(t) = -i \frac{\mu_B B'}{\hbar} e^{-i(\omega_0 - \omega)t} C_\uparrow(t) \end{cases}$$

This is a system of first-order differential equations, which I can transform into a single second-order ordinary differential equation. Let me make the replacement $V = 2\mu_B B'$ (V can be thought of as being the energy gap between levels due to the perturbation only).

Let us take the time derivative of ~~Ⓑ~~: Ⓐ

$$Ⓒ \quad \ddot{C}_\uparrow = i(\underset{\text{derivative of exponential}}{\omega_0 - \omega}) \dot{C}_\uparrow \underbrace{- i \frac{V}{2\hbar} e^{i(\omega_0 - \omega)t} \dot{C}_\downarrow(t)}_{\text{I can write this as } \dot{C}_\uparrow / C_\downarrow}$$

Take Ⓐ and multiply it by Ⓑ, side by side

$$\dot{C}_\uparrow \dot{C}_\downarrow = -\frac{V^2}{4\hbar^2} C_\downarrow C_\uparrow$$

We finally obtain (note the sign flip : $i(\omega_0-\omega) = -i(\omega-\omega_0)$) :

$$\ddot{C}_\uparrow = -i(\omega-\omega_0)\,\dot{C}_\uparrow - \frac{V^2}{4\hbar^2}\,C_\uparrow$$

This is the same equation as a LCR circuit, a mass on a spring with friction : it is a damped oscillator. The solution has the form $e^{\lambda t}$, where λ are the solution of the algebraic equation :

$$\lambda^2 + i(\omega-\omega_0)\lambda + V^2/4\hbar^2 = 0$$

Solution : $$\lambda = \frac{-i(\omega-\omega_0) \pm \sqrt{-(\omega_0-\omega)^2 - V^2/\hbar^2}}{2} = i\left(-\frac{\omega-\omega_0}{2} \pm \omega_R\right)$$

$$\omega_R = \frac{1}{2}\sqrt{(\omega-\omega_0)^2 + V^2/\hbar^2}$$

RABI FLOPPING FREQUENCY
(if V is complex, use $|V|^2$ here)

Let us massage a bit our solution :

$$C_\uparrow(t) = e^{\lambda t} = e^{-i\frac{\omega-\omega_0}{2}t}\left(a e^{i\omega_R t} + b e^{-i\omega_R t}\right)$$

a and b are fixed by initial conditions. Let us say that we were, at $t=0$, in $|\downarrow\rangle$. Then: $C_\uparrow(0)=0$ and $C_\downarrow(0)=1 \Rightarrow$

$$C_\uparrow(0) = 0 \implies a = -b$$

$$C_\downarrow(0) = 1 = \underbrace{i\frac{2\hbar}{V}e^{i\omega_0 t}\cdot\dot{C}_\uparrow(0)}_{\text{use equation (A)}} = \underbrace{i\frac{2\hbar}{V}\cdot 1\cdot\left(-i\left(\frac{\omega-\omega_0}{2}-\omega_R\right)\cdot a - i\left(\frac{\omega-\omega_0}{2}+\omega_R\right)b\right)}_{\text{take derivative before setting } t=0\,!}$$

$$\underset{b=-a}{=} i\frac{2\hbar}{V}\cdot(-i)(-2\omega_R)a \implies a = -b = \frac{-V}{4\hbar\omega_R}$$

One last operation : $$a e^{i\omega_R t} + b e^{-i\omega_R t} \underset{b=-a}{=} a\left(e^{i\omega_R t} - e^{-i\omega_R t}\right) = 2ia\sin(\omega_R t).$$

We can finally write the full, exact solution :

$$C_\downarrow(t) = e^{i(\omega-\omega_0)t/2}\left(\cos(\omega_r t) + i\,\frac{\omega_0-\omega}{2\omega_r}\sin(\omega_r t)\right)$$

$$C_\uparrow(t) = -\frac{iV}{2\hbar\omega_r}\,e^{i(\omega_0-\omega)t/2}\,\sin(\omega_r t)$$

Let us test if it makes sense :

1)- at $t=0$: $C_\downarrow(0) = 1$, $C_\uparrow(0) = 0$: great

2)- for weak driving conditions, $V \ll \hbar(\omega-\omega_0)$, it should match our approximation. Let us check this limit :

$$\omega_r = \frac{1}{2}\sqrt{(\omega-\omega_0)^2 + V^2/\hbar^2} \approx \frac{\omega-\omega_0}{2}$$

$$C_\uparrow(t) \approx -\frac{iV}{\hbar(\omega-\omega_0)}\,e^{i(\omega_0-\omega)t/2}\,\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

$$|C_\uparrow(t)|^2 \approx \frac{V^2}{\hbar^2}\,\frac{\sin^2\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega-\omega_0)^2}$$: exactly our first-order solution
(almost: factor of 2^2 difference: here we have both B_x and B_y, before just B_x!

3)- we now have a solution also for $\omega = \omega_0$, where our first-order solution fails ($P(\omega) \propto t^2$) : $\omega_r = V/2\hbar$

$|C_\uparrow(t)|^2 = \sin^2(\omega_r t)$ <u>now</u> probabilities are bounded by 1
$|C_\downarrow(t)|^2 = \cos^2(\omega_r t)$