TIME-DEPENDENT PERTURBATION THEORY: EXAMPLE

Let us imagine to have a spin-1/2 particle, at rest, in a static magnetic field $\vec{B} = B_z \hat{z}$. The Hamiltonian H^0 will be:

$$H^0 = -\vec{\mu} \cdot \vec{B} \qquad \text{where } \vec{\mu} = g\mu_B \vec{S}/\hbar$$

Let us now use a spinor notation in which $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\vec{S} \propto \vec{\sigma}$ (σ = Pauli matrices). We can write:

$$H^0 = \mu_B B_z \sigma_z = \begin{pmatrix} \mu_B B_z & 0 \\ 0 & -\mu_B B_z \end{pmatrix}$$

The eigenstates of this unperturbed Hamiltonian are $\psi_a = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, energy $-\mu_B B_z$, and $\psi_b = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, energy $+\mu_B B_z$.

Let us now add a perturbation in the form of a varying field along $\hat{x}$:

$$\vec{B}' = B_x(t)\,\hat{x} \Longrightarrow H'(t) = \mu_B B_x(t)\,\sigma_x$$

Note that $H' = \begin{pmatrix} 0 & \mu_B B_x(t) \\ \mu_B B_x(t) & 0 \end{pmatrix}$, hence $\langle \psi_a | H' | \psi_a \rangle = \langle \psi_b | H' | \psi_b \rangle = 0$, a condition we assumed as usually valid.

Let us assume that $B_x(t) = B_x \cos(\omega t)$, an oscillating field, and that at time $t=0$, when we turn on the perturbation, we are in the state $|\downarrow\rangle$. That is, at $t=0$, $c_\downarrow = 1$ and $c_\uparrow = 0$.

We saw that:

$$\dot{c}_\uparrow(t) = -\frac{i}{\hbar} \langle \uparrow | H' | \downarrow \rangle e^{i\frac{E_\uparrow - E_\downarrow}{\hbar}t} c_\downarrow(t) \approx -\frac{i}{\hbar} \langle \uparrow | H' | \downarrow \rangle e^{2i\mu_B B_z t/\hbar}$$

Why? At <u>zeroth order</u>, $c_\downarrow(t) = 1$ CONSTANT: the Hamiltonian is just H^0, and if I start in $|\downarrow\rangle$, I stay there. Expression above is therefore valid only at <u>first-order</u>.

Hence, we get (again, <u>only</u> at first-order):

$$C_\uparrow(t) = -\frac{i}{\hbar}\int_0^t e^{2i\mu_B B_z t'/\hbar}\cdot \underbrace{\mu_B B_x \cos\omega t'}\, dt'$$

$$\langle\uparrow|H'|\downarrow\rangle = (1\ 0)\begin{pmatrix} 0 & \mu_B B_x(t) \\ \mu_B B_x(t) & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \mu_B B_x(t)$$

This number is in general $\neq 0$, meaning that my oscillating $B_x(t)$ field can induce transitions from a spin $\downarrow$ state to a spin $\uparrow$ state.

Let me define $\omega_0 = 2\mu_B B_z/\hbar$, and use $\cos\omega t = \dfrac{e^{i\omega t}+e^{-i\omega t}}{2}$:

$$C_\uparrow(t) = -\frac{i}{2\hbar}\mu_B B_x\int_0^t [e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}]\, dt' =$$

$$= -\frac{\mu_B B_x}{2\hbar}\cdot\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]$$

Here I think about the case $\omega \approx \omega_0$. What is it? I am driving the $B_x(t)$ field at the same frequency corresponding to the energy difference between the two possible states of my system. It is a rather interesting case. For $\omega\approx\omega_0$, the first term becomes negligible ($\frac{1}{\omega_0+\omega} \ll \frac{1}{\omega_0-\omega}$). Hence:

$$C_\uparrow(t) \approx -\frac{\mu_B B_x/2\hbar}{\omega_0-\omega}\cdot e^{i(\omega_0-\omega)t/2}\ \frac{e^{i(\omega_0-\omega)t/2}-e^{-i(\omega_0-\omega)t/2}}{2i}\cdot 2i$$

$$= -\frac{i\mu_B B_x}{\hbar}\ \frac{\sin((\omega_0-\omega)t/2)}{\omega_0-\omega}\ e^{i(\omega_0-\omega)t/2}$$

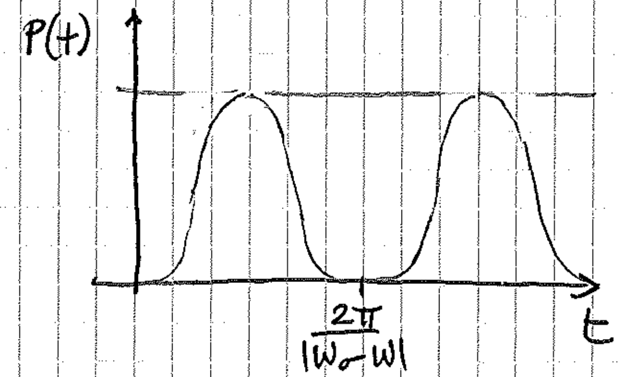
Finally:

$$|C_\uparrow(t)|^2 = \left(\frac{\mu_B B_x}{\hbar}\right)^2 \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2} = \text{probability, at first-order approx.,}$$

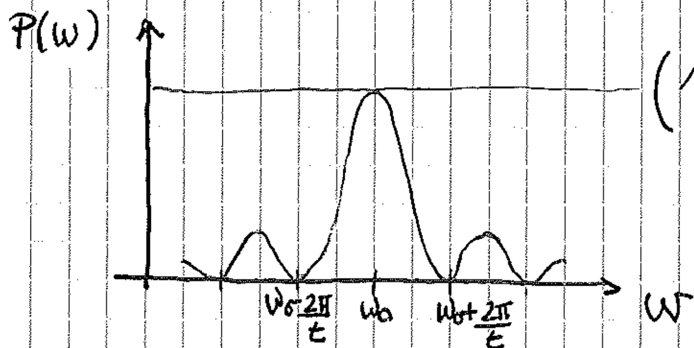
that, starting in state $\downarrow$, I find myself in state $\uparrow$ after time t.

$$P_{\downarrow\to\uparrow}(t) \underset{\substack{\text{first order}\\ \text{approx}}}{\approx} \left(\frac{\mu_B B_x}{\hbar}\right)^2 \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

How does this look? Remember we are doing first-order approximations



$$\left(\frac{\mu_B B_x}{\hbar(\omega_0-\omega)}\right)^2 << 1 \quad : \text{I need } H' << H^0 !$$



$$\left(\frac{\mu_B B_x t}{2\hbar}\right)^2 << 1$$

Points to note:

- the probability of finding the system in state $\uparrow$ <u>oscillates</u> with frequency determined by the detuning $\Delta\omega = |\omega_0 - \omega|$, where ω is the frequency of the perturbation (the "driving" force!) and $\omega_0 = (E_\uparrow - E_\downarrow)/\hbar$ is the energy difference between the $\uparrow$ and $\downarrow$ states (the "natural" frequency of my system!)
- the probability increases with the square of the driving force, B_x, but the result is valid only if perturbation theory applies ($H' << H^0$)
- $P(\omega)$ peaks at $\omega = \omega_0$: when driving at natural frequency, P becomes:
  $$\lim_{\omega\to\omega_0} P_{\downarrow\to\uparrow}(t) = (\mu_B B_x/\hbar)^2 \cdot \left(\frac{t}{2}\right)^2$$
  i.e., the probability of a transition increases quadratically with time, until the point where perturbation theory is not valid any more