EXCHANGE FORCES

Let us imagine to have a system of two particles in states (a) and (b), respectively. The 2-particle wavefunction will be:

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2)$$

If the two particles are indistinguishable, relativity and spin-statistic theorem tell us that we have two possiblities:

1 and 2 are identical fermions (semi-integer spin): $\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right)$

1 and 2 are identical bosons (integer spin): $\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right)$

Let us look at the average distance between the two particles:

$$\langle |\vec{r}_1 - \vec{r}_2|^2 \rangle = \iint d^3\vec{r}_1\, d^3\vec{r}_2 \;\psi^*(\vec{r}_1, \vec{r}_2)\; |\vec{r}_1 - \vec{r}_2|^2\, \psi(\vec{r}_1, \vec{r}_2)$$

Let us also move to a 1-D world to simplify the notation: $\vec{r} \to x$

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

$$\langle x_1^2 \rangle = \iint dx_1 dx_2 \,\frac{1}{\sqrt{2}}\left(\psi_a^*(x_1)\psi_b^*(x_2) \pm \psi_a^*(x_2)\psi_b^*(x_1)\right) x_1^2 \,\frac{1}{\sqrt{2}}\left(\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\right)$$

$$= \iint dx_1 dx_2 \,\frac{1}{2} |\psi_a(x_1)|^2 |\psi_b(x_2)|^2 x_1^2 \; +$$

$$+ \iint dx_1 dx_2 \,\frac{1}{2} |\psi_a(x_2)|^2 |\psi_b(x_1)|^2 x_1^2 \; +$$

$$\pm \iint dx_1 dx_2 \,\frac{1}{2} \psi_a^*(x_1)\psi_a(x_2)\, x_1^2\, \psi_b^*(x_2)\,\psi_b(x_1) \; +$$

$$\pm \iint dx_1 dx_2 \,\frac{1}{2} \psi_a^*(x_2)\,\psi_a(x_1)\, x_1^2\, \psi_b^*(x_1)\,\psi_b(x_2)$$

All good so far, but we saw that, at least in the case of fermions, $\psi_a(\vec{x}) \neq \psi_b(\vec{x})$, otherwise $\psi(\vec{r}_1, \vec{r}_2) = 0$ (Pauli's exclusion principle).
Let us then take our two identical particles to be in two different and orthogonal states. Typical situation: two electrons in different orbital states. Then:

$$\langle x_1^2 \rangle = \frac{1}{2} \left(\langle x_1^2 \rangle_a + \langle x_1^2 \rangle_b \right)$$

Here's how (back to page before):

$$\langle x_1^2 \rangle_a = \int dx_1 |\psi_a|^2 x_1^2 \underbrace{\int dx_2 |\psi_b(x_2)|^2}_{1}$$

$$\langle x_1^2 \rangle_b = \underbrace{\int dx_2 |\psi_a(x_2)|^2}_{1} \int dx_1 |\psi_b(x_1)|^2 x_1^2$$

$$\int dx_2 \; \psi_b^*(x_2) \psi_a(x_2) = 0 \implies \text{third term in sum} = 0$$

$$\int dx_2 \; \psi_a^*(x_2) \psi_b(x_2) = 0 \implies \text{fourth term in sum} = 0$$

The same calculation must apply for $\langle x_2^2 \rangle$, after all the two particles are indistinguishable. What about $\langle x_1 x_2 \rangle$?

$$\langle x_1 x_2 \rangle = \frac{1}{2} \int dx_1 \int dx_2 \left(\psi_a^*(x_1) \psi_b^*(x_2) \pm \psi_a^*(x_2) \psi_b^*(x_1) \right) x_1 x_2 \left(\psi_a(x_1) \psi_b(x_2) \pm \psi_a(x_2) \psi_b(x_1) \right)$$

$$= \frac{1}{2} \int dx_1 \psi_a^*(x_1) x_1 \psi_a(x_1) \int dx_2 \psi_b^*(x_2) x_2 \psi_b(x_2) +$$

$$+ \frac{1}{2} \int dx_1 \psi_b^*(x_1) x_1 \psi_b(x_1) \int dx_2 \psi_a^*(x_2) x_2 \psi_a(x_2) +$$

$$\pm \frac{1}{2} \int dx_1 \psi_a^*(x_1) x_1 \psi_b(x_1) \int dx_2 \psi_b^*(x_2) x_2 \psi_a(x_2) +$$

$$\pm \frac{1}{2} \int dx_1 \psi_b^*(x_1) x_1 \psi_a(x_1) \int dx_2 \psi_a^*(x_2) x_2 \psi_b(x_2)$$

$$2\langle x_1 x_2\rangle = \langle x_1\rangle_a \langle x_2\rangle_b + \langle x_2\rangle_a \langle x_1\rangle_b +$$

$$\pm \langle x_1\rangle_{ab} \langle x_2\rangle_{ba} \pm \langle x_2\rangle_{ab} \langle x_1\rangle_{ba}$$

Two notes: 1) $\langle x_1\rangle_a = \langle x_2\rangle_a$: indistinguishable particles
$\langle x_1\rangle_b = \langle x_2\rangle_b$

2) $\langle x\rangle_{ab} = \langle x\rangle_{ba}^*$: check integral definition

Therefore, putting everything together:

$$\langle x_1 x_2\rangle = \langle x\rangle_a \langle x\rangle_b \pm |\langle x\rangle_{ab}|^2$$

Finally:

$$\langle |x_1 - x_2|^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2$$

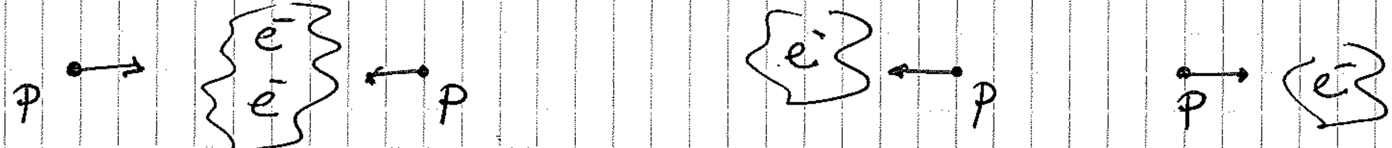
Bosons feel an effective "exchange force" pushing them together, while fermions feel an effective "exchange force" pushing them apart:

$$\langle |x_1 - x_2|^2\rangle_{bosons} < \langle |x_1 - x_2|^2\rangle_{fermions}$$

Exchange force is purely geometrical consequence of symmetry requirement. Let us consider a H_2 molecule. Let us consider what happens if our electrons (a pair of identical fermions) are in a symmetric spatial configuration, or in an anti-symmetric spatial configuration:

symmetric: they tend to stay close to each other

anti-symmetric: exchange force tends to push them apart



molecule is bound together

molecule is torn apart

Did we just prove that the H_2 molecule cannot exist?

Hard to believe. The fact is that we need to consider the complete state of the system, which includes spin: the state of an electron is $\psi(\vec{r}) \cdot \chi(\vec{S})$ [technically, S_z only: I know $S^2 = 3\hbar^2/4$].

Then, my 2-electron state can be written in such a way that it is spatially symmetric and spin anti-symmetric. The total state becomes anti-symmetric, and it is acceptable for a pair of identical fermions.

How do I write an anti-symmetric spin state?

$$\frac{\chi_1(+\hbar/2)\chi_2(-\hbar/2) - \chi_1(-\hbar/2)\chi_2(+\hbar/2)}{\sqrt{2}} = \frac{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle}{\sqrt{2}}$$

$$= |0\ 0\rangle \quad S_z = S_{z1} + S_{z2}, \quad \vec{S} = \vec{S}_1 + \vec{S}_2$$

Indeed, chemists measure that the two electrons in a H_2 molecule are in a state with total spin $= 0$.

Example of exchange force: neutron stars. Very heavy (estimate density of neutrons tightly together to density of matter, in which $r_{proton} \sim 1\ fm = 10^{-15}\ m$ and $r_{atom} \sim 1\ Å = 10^{-10}\ m$), the gravitational pull towards collapse is balanced by the exchange force: neutrons are fermions and don't like being pulled too close together.