IDENTICAL PARTICLES

Let us assume that we encounter the following problem: describe a 2-particle system. It is natural to extend our 1-particle wavefunction $\psi(\vec{r}, t)$ to a 2-particle wavefunction $\psi(\vec{r}_1, \vec{r}_2, t)$.

The Schroedinger's equation becomes:

$$H\psi(\vec{r}_1, \vec{r}_2, t) = i\hbar \frac{\partial \psi(\vec{r}_1, \vec{r}_2, t)}{\partial t}$$

where:

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$$

this Laplacian operates only on $\vec{r}_2$

mass of particle 2

The statistical interpretation of the wavefunction is the same:

$|\psi(\vec{r}_1, \vec{r}_2, t)\, d^3\vec{r}_1 d^3\vec{r}_2|^2$ = probability of finding particle 1 in volume $d^3\vec{r}_1$ and particle 2 in volume $d^3\vec{r}_2$ at time t

Normalization:

$$\int d^3\vec{r}_1 \int d^3\vec{r}_2 \; |\psi(\vec{r}_1, \vec{r}_2, t)|^2 = 1 \qquad \text{for any } t$$

In the special case in which V, the potential, does <u>NOT</u> depend on time, I can write:

$$\psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2)\, e^{-iEt/\hbar}$$

where $\psi(\vec{r}_1, \vec{r}_2)$ satisfies:

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi(\vec{r}_1, \vec{r}_2) - \frac{\hbar}{2m_2}\nabla_2^2\psi(\vec{r}_1, \vec{r}_2) + V\psi(\vec{r}_1, \vec{r}_2) = E\psi(\vec{r}_1, \vec{r}_2)$$

i.e., the usual time-independent Schroedinger's equation.

Let us now suppose that particle-1 is in the 1-particle state $\psi_a(\vec{r}_1)$, and particle-2 in the 1-particle state $\psi_b(\vec{r}_2)$. Then:

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)$$

Let us put aside the fact that the expression above is not telling the whole story (think about entangled states) and assume for now that it is correct. Still, it assumes that I can tell which particle is which. This is not so trivial in quantum mechanics: electrons are <u>identical</u>, I cannot attach a label that I can observe and can tell me which electron is electron-1 and which electron is electron-2. Therefore, $\psi(\vec{r}_1, \vec{r}_2)$ MUST look identical if I exchange the labels (a) and (b). There are only two possibilities:

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)$$

SPIN-STATISTIC THEOREM: this rule is constraint from relativity!

We can distinguish two types of particles:

- bosons: use the symmetric $\psi(\vec{r}_1, \vec{r}_2)$; their spin is integer
- fermions: use the anti-symmetric $\psi(\vec{r}_1, \vec{r}_2)$; their spin is half-integer

E.g.: bosons: photon (spin=1), Higgs (spin=0)

fermions: muon, electron, quarks (spin = 1/2), Ω^- (spin = 3/2)

better yet: exchange operator

Let us introduce the parity operator P:

$$P\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

I.e., P swaps particle-1 and particle-2. If I apply P again, I get back to $\psi(\vec{r}_1, \vec{r}_2)$:

$P^2 = \mathbb{1}$ identity operator.

Therefore, if $\psi(\vec{r}_1, \vec{r}_2)$ is an eigenfunction of P, its eigenvalue can only be either 1 or -1: $\psi = \overset{= \mathbb{1} \text{ identity operator}}{P^2} \psi = P(P\psi) = P(\lambda\psi) = \lambda P \psi = \lambda^2 \psi$

$\Rightarrow \lambda^2 = 1$.

If two particles are identical, then it must be that $[P, H] = 0$: the Hamiltonian must commute with P because I cannot be able to tell particle-1 from particle-2: hence, I must have $m_1 = m_2$ and $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_2, \vec{r}_1)$

We can then choose solutions of the Hamiltonian which are also eigenstates of P. There are two types:

P eigenvalue

$\lambda = 1$: $\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$: symmetric states, we use these for identical bosons

$\lambda = -1$: $\psi(\vec{r}_1, \vec{r}_2) = -\psi(\vec{r}_2, \vec{r}_1)$: antisymmetric states, used for identical fermions

Going back to $\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)$, we now see clearly that:

$$P\left(\psi_a(\vec{r}_1)\psi(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right) = \pm\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right)$$

i.e., $+$ = symmetric solutions, bosons, $P\psi = +\psi$; $-$: antisymmetric solutions, fermions, $P\psi = -\psi$.

What if both particles are in the <u>same</u> state $\psi_a(\vec{r})$? I.e., what if $\psi_b(\vec{r}) = \psi_a(\vec{r})$?

Bosons: $\psi(\vec{r}_1, \vec{r}_2) = \underset{\text{can remove normalizing}}{2}\, \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)$; all good: $P\psi = \psi$

Fermions: $\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_a(\vec{r}_1) = 0$!

This is <u>Pauli's exclusion principle</u>: I cannot have a system of two (or more: we will generalize soon) identical fermions in the same state.

<u>Example</u>: 1-D quantum well containing two particles with same mass

The one-particle states are known:

$$\psi_n(x_1) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}$$

a) Two distinguishable particles:

ground state: $\psi(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

$$E = \frac{\pi^2\hbar^2}{ma^2}$$

first excited state: $\psi^{(1)}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

$$\psi^{(2)}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)$$

$$E^{(1)} = E^{(2)} = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$$

the first excited state is degenerate, $\psi^{(1)}$ and $\psi^{(2)}$ are orthogonal (test!) and are eigenstates of H with the same eigenvalue

b) indistinguishable <u>bosons</u>:

ground state: $\psi(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

$$E = \frac{\pi^2\hbar^2}{ma^2}$$

no changes: $\psi(x_1,x_2) = \psi(x_2,x_1)$, it is good for bosons

first excited state: $\psi(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_2}{a}\right)\sin\left(\frac{\pi x_1}{a}\right)\right]$

$$E = \frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$$

I need a symmetric state, and there is only one with energy $\frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$:

the first excited state is no longer degenerate, if my system is composed of a pair of indistinguishable bosons.

c) indistinguishable <u>fermions</u>:

(former) ground state, anti-symmetrized:

$$\psi(x_1,x_2) \propto \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)-\sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{\pi x_1}{a}\right) = \underline{\underline{0}}$$

if my system is composed of two identical fermions, it cannot have energy $\frac{\pi^2\hbar^2}{ma^2}$ because I cannot have both fermions in the same state

(former) first excited state, anti-symmetrized, now ground state:

as in lowest level

$$\psi(x_1,x_2)=\frac{\sqrt{2}}{a}\left(\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)-\sin\left(\frac{2\pi x_2}{a}\right)\sin\left(\frac{\pi x_1}{a}\right)\right)$$

$$E=\frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}$$