REVIEW

ADDITION OF ANGULAR MOMENTA

We often encounter the situation in which a system's angular momentum has more than one contribution. E.g.: an electron in an atom has both orbital angular momentum and intrinsic angular momentum (its spin); a two-particle system can be described using the individual particle spins $\vec{S}^{(1)}$ and $\vec{S}^{(2)}$, or their sum. How does it work? Let us take the case of an electron with spin ($= 1/2$) and orbital angular momentum ℓ (i.e., $L^2 \psi = \hbar^2 \ell(\ell+1)\psi$)

(all works even if spin $\neq 1/2$ ...)

$$\vec{J} = \vec{L} + \vec{S}$$

uncoupled notation: $|\ell\ m\rangle |s\ m_s\rangle$

coupled notation: $|j\ m_j\rangle$

Defining feature: uncoupled notation diagonalizes L^2, L_z, S^2, S_z, while coupled notation diagonalizes J^2, J_z:

(and also L^2 and S^2, since they both commute with J^2 and J_z)

$$L^2 |\ell m\rangle |s\ m_s\rangle = \hbar\, \ell(\ell+1)\, |\ell m\rangle |s\ m_s\rangle$$
$$L_z |\ell m\rangle |s\ m_s\rangle = \hbar\, m\, |\ell m\rangle |s\ m_s\rangle$$
$$S^2 |\ell m\rangle |s\ m_s\rangle = \hbar\, s(s+1)\, |\ell m\rangle |s\ m_s\rangle$$
$$S_z |\ell m\rangle |s\ m_s\rangle = \hbar\, m_s\, |\ell m\rangle |s\ m_s\rangle$$

$$J^2 |j\ m_j\rangle = \hbar\, j(j+1)\, |j\ m_j\rangle$$
$$J_z |j\ m_j\rangle = \hbar\, m_j\, |j\ m_j\rangle$$

We can move from coupled to uncoupled (and viceversa) using Clebsch-Gordan coefficients. Important to remember (gives a sanity check!):

$|\ell m\rangle |s\ m_s\rangle$ can be written as combination of $|j\ m_j\rangle$ states such that

$\boxed{j = \ell+s, \ldots, |\ell - s|}$ and $\boxed{m_j = m + m_s}$

Example :

$1 \times 1/2$ ($\ell_1 \times \ell_2$)

m_1	m_2	$j = 3/2$, $m_j = +1/2$	$j = 1/2$, $m_j = +1/2$
$+1$	$-1/2$	$1/3$	$2/3$
0	$+1/2$	$2/3$	$-1/3$

from uncoupled to coupled: read horizontal rows:

$$|\ell_1 m_1\rangle |\ell_2 m_2\rangle = \sum_j C^{\ell_1 \ell_2 j}_{m_1 m_2 m_j} |j\ m_j\rangle$$

first row:

$$|1\ 1\rangle |1/2\ -1/2\rangle = \sqrt{1/3}\ |3/2\ \ 1/2\rangle + \sqrt{2/3}\ |1/2\ \ 1/2\rangle$$

second row:

$$|1\ 0\rangle |1/2\ +1/2\rangle = \sqrt{2/3}\ |3/2\ \ 1/2\rangle - \sqrt{1/3}\ |1/2\ \ 1/2\rangle$$

from coupled to uncoupled: read vertical columns:

$$|j\ m_j\rangle = \sum_{m_1+m_2=m_j} C^{\ell_1 \ell_2 j}_{m_1 m_2 m_j} |\ell_1\ m_1\rangle |\ell_2\ m_2\rangle$$

first column:

$$|3/2\ \ 1/2\rangle = \sqrt{1/3}\ |1\ 1\rangle |1/2\ -1/2\rangle + \sqrt{2/3}\ |1\ 0\rangle |1/2\ \ 1/2\rangle$$

second column:

$$|1/2\ \ 1/2\rangle = \sqrt{2/3}\ |1\ 1\rangle |1/2\ -1/2\rangle - \sqrt{1/3}\ |1\ 0\rangle |1/2\ \ 1/2\rangle$$

(j, m_j; ℓ_1, m_1; ℓ_2, m_2)

PERTURBATION THEORY, TIME-INDEPENDENT

$H^0 \psi_m^0 = E_m^0 \psi_m^0$: ψ_m^0 are eigenstates of UNPERTURBED Hamiltonian
the corresponding energy eigenvalue is E_m^0

Let us add a perturbation H': how can I approximate E_m and ψ_m, eigenvalues and eigenstates of $H = H^0 + H'$?

first order

$$E_m \cong E_m^0 + E_m^1$$
$$\psi_m \cong \psi_m^0 + \psi_m^1$$

$$\boxed{E_m^1 = \langle \psi_m^0 | H' | \psi_m^0 \rangle}$$

$$\boxed{\psi_m^1 = \sum_{n \neq m} \frac{\langle \psi_n^0 | H' | \psi_m^0 \rangle}{E_m^0 - E_n^0} \psi_n^0}$$

method is valid only if $\boxed{|E_m^1| \ll |E_m^0|}$

what if more than one ψ_m^0 state is an eigenstate of H^0 corresponding to the same E_m^0? Let us call these states $\psi_{(1)}^0, \psi_{(2)}^0 \dots \psi_{(p)}^0$; they satisfy the following condition:

$$H^0 \psi_{(1)}^0 = E^0 \psi_{(1)}^0$$
$$\vdots \qquad \text{with the same } E^0$$
$$H^0 \psi_{(p)}^0 = E^0 \psi_{(p)}^0$$

I put parentheses to clearly indicate that this index has a different meaning than the n we used above.

Let us build the W matrix:

$$\boxed{W_{ij} = \langle \psi_{(i)}^0 | H' | \psi_{(j)}^0 \rangle}$$

and solve its eigenvector problem: $W\vec{\alpha} = E^1 \vec{\alpha}$ (linear algebra problem!)
W is $p \times p$ matrix $\Rightarrow$ I will find p eigenvectors $\vec{\alpha}_{(q)}$ and p eigenvalues $E_{(q)}^1$. The eigenvectors tell me which are the "good" linear combinations of $\psi_{(p)}^0$ states that get a energy correction $E_{(p)}^1$.

Let $\vec{\alpha}_{(1)}, \ldots \vec{\alpha}_{(p)}$ be the eigenvectors of W, and $E^1_{(1)} \ldots E^1_{(p)}$ their corresponding eigenvalues. Then:

$$(H^0 + H')\sum_i \alpha_{(1)i}\,\psi^0_{(i)} \cong (E^0 + E^1_{(1)})\sum_i \alpha_{(1)i}\,\psi^0_{(i)}$$

$$\vdots$$

$$(H^0 + H')\sum_i \alpha_{(p)i}\,\psi^0_{(i)} \cong (E^0 + E^1_{(p)})\sum_i \alpha_{(p)i}\,\psi^0_{(i)}$$

The p linear combinations $\sum_i \alpha_{(1)i}\,\psi^0_{(i)} \ldots \sum_i \alpha_{(p)i}\,\psi^0_{(i)}$ are the "good" linear combinations.

We have p eigenvectors $\vec{\alpha}_{(i)}$, and each of them has p components: think of them as a normal-mode expansion of a beaded string with p masses: each normal mode (and I have p of them) identifies a vector with p components, each component being the amplitude of the oscillation of the p-th bead.

ANGULAR MOMENTA AND MAGNETIC FIELDS

$H = -\vec{\mu}_{TOTAL} \cdot \vec{B}$ contribution to Hamiltonian due to interaction between magnetic field $\vec{B}$ and magnetic moment $\vec{\mu}$

$\vec{\mu}_{spin} = \mu_B \cdot g \frac{\vec{S}}{\hbar}$ $g = -2$ for electron (assume $g = 1$, unless otherwise specified)

(-2: effect of electron charge)

$\vec{\mu}_{orbital} = -\mu_B \cdot \vec{L}/\hbar$

(minus: electron is negatively charged)

e.g. $g_{proton} = +5.6$

$g_{neutron} = -3.8$

$g_{muon} = -2$