DEGENERATE PERTURBATION THEORY (TIME-INDEPENDENT)

<u>Example</u>

Let us take a 2D oscillator:

$$H^0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}kx^2 + \frac{1}{2}ky^2$$

The ground state is non-degenerate; its energy is $\hbar\omega$.

Let us indicate with $|n, m\rangle$ the eigenstates of H^0, with eigenvalues $\hbar\omega(n+m+1)$. $|1,0\rangle$ and $|0,1\rangle$ (1^{st} excitation along x and along y, respectively) both have the same energy: $2\hbar\omega \Rightarrow$ <u>degeneracy</u>

(in general, $|n,m\rangle$ states have a degeneracy $p = n+m+1$: $|4,5\rangle, |5,4\rangle, |9,0\rangle, |1,8\rangle \dots$ all have the same energy)

Let us introduce the (usual) notation:

a, b : ladder operators for the x and y components

$|n, m\rangle$: n-th excited states along x component and m-th excited state along the y component

We have:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+) \qquad a\,|n,m\rangle = \sqrt{n}\,|n-1, m\rangle$$
$$a^+\,|n,m\rangle = \sqrt{n+1}\,|n+1, m\rangle$$
$$y = \sqrt{\frac{\hbar}{2m\omega}}\,(b + b^+) \qquad b\,|n,m\rangle = \sqrt{m}\,|n, m-1\rangle$$
$$b^+\,|n,m\rangle = \sqrt{m+1}\,|n, m+1\rangle$$

Let us now introduce the perturbation:

$$H' = k'\,xy$$

and evaluate its effect on the first excited state, which is degenerate, with a two-fold degeneracy:

$E^0 = 2\hbar\omega$; both $|1,0\rangle$ and $|0,1\rangle$ have the same energy

Let us build the W matrix:

$$W = \begin{pmatrix} \langle 1,0|H'|1,0\rangle & \langle 1,0|H'|0,1\rangle \\ \langle 0,1|H'|1,0\rangle & \langle 0,1|H'|0,1\rangle \end{pmatrix}$$

$$H' = K'xy = K'\left(\frac{\hbar}{2m\omega}\right)(a+a^+)(b+b^+) = \left(\frac{\hbar}{2m\omega}\right)K'\,(ab + ab^+ + a^+b + a^+b^+)$$

[note: a and b commute because they operate on different components)

Then:

$\langle 1,0|H'|1,0\rangle = \emptyset$: no term in H' leaves $|1,0\rangle$ unchanged: it either becomes $|0,0\rangle$ or $|1,1\rangle$

$\langle 0,1|H'|0,1\rangle = \emptyset$: same reasoning

$$\langle 1,0|H'|0,1\rangle = \left(\frac{\hbar}{2m\omega}\right)K'\,\langle 1,0|\,a^+b\,|0,1\rangle = \left(\frac{\hbar}{2m\omega}\right)K'$$

$$\langle 0,1|H'|1,0\rangle = \left(\frac{\hbar}{2m\omega}\right)K'\,\langle 0,1|\,ab^+\,|1,0\rangle = \left(\frac{\hbar}{2m\omega}\right)K'$$

only this term transforms $|1,0\rangle$ into $|0,1\rangle$

Finally:

$$W = \left(\frac{\hbar}{2m\omega}\right)K'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Diagonalization is straightforward (solve linear algebra problem: $\overset{\text{matrix}}{W}\underset{\text{vector}}{\vec{\alpha}} = \overset{\text{numbers}}{E}\vec{\alpha}$)

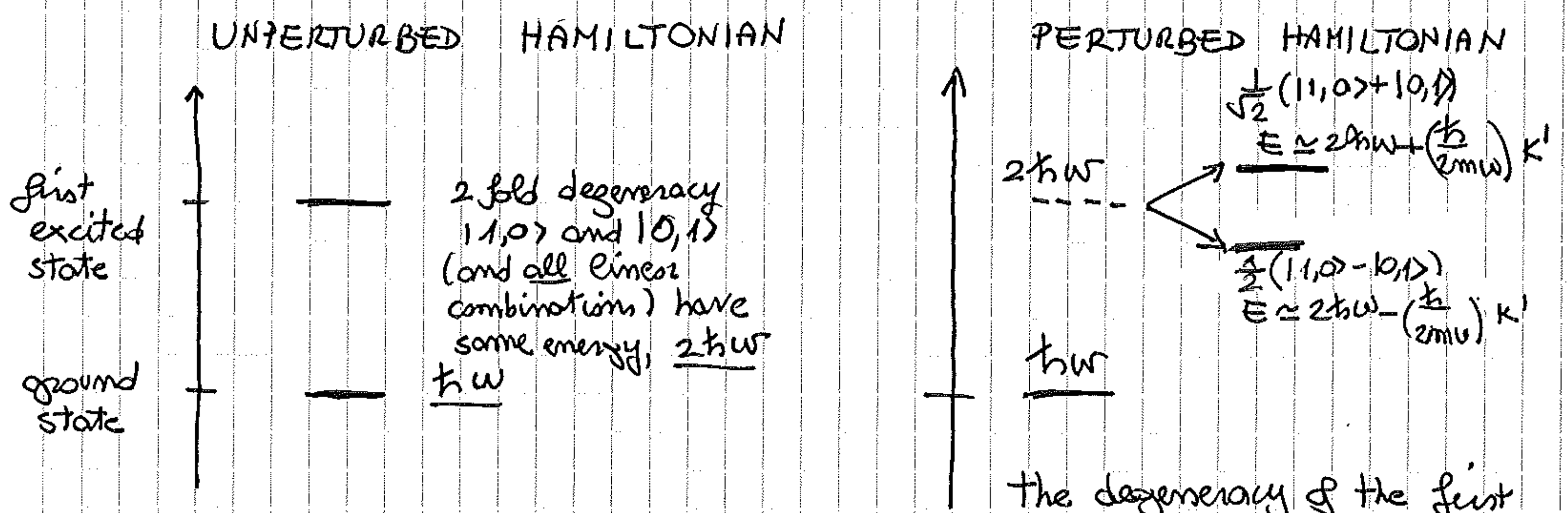
$$\vec{\alpha}_{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad E_{(1)} = \left(\frac{\hbar}{2m\omega}\right)K'$$

$$\vec{\alpha}_{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad E_{(2)} = -\left(\frac{\hbar}{2m\omega}\right)K'$$

What is the meaning of this result? The meaning of it is that the "good" eigenstates of the perturbed Hamiltonian are <u>not</u> $|1,0\rangle$ and $|0,1\rangle$ but rather:

$$\frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right) \quad \text{and} \quad \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right)$$

Here is a picture of what is happening:



the degeneracy of the first excited state is lifted