Physics 402
Spring 2019
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Formula Sheet Final Exam

Infinite square well

$\mathcal{H}^0 = \frac{p^2}{2m} + V(x) \qquad V(x) = \begin{cases} 0 \text{ for } 0 < x < a \\ \infty \text{ for x<0 and x>a} \end{cases}$

$E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$ and $\psi^0(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \text{ for } 0 < x < a \\ 0 \text{ for x<0 and x>a} \end{cases}$ with $n = 1, 2, 3, \ldots$

Harmonic Oscillator

$H^0 = \frac{p_x^2}{2m} + \frac{k}{2}x^2$, $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$, $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, $E_n = (n + \frac{1}{2})\hbar\omega$; $n = 0, 1, 2, \ldots$;

$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$; $p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$;

$a_+\psi_n = \sqrt{n+1}\psi_{n+1}$; $a_-\psi_n = \sqrt{n}\psi_{n-1}$

Hydrogen Atom

$\mathrm{H}^0 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{(-e)(+e)}{4\pi\varepsilon_0 r}$, $n = 1, 2, 3, \ldots, \ell = 0, 1, \ldots, n-1$; $-\ell \leq m \leq \ell$,

$\psi_{n,\ell,m}^0(r,\theta,\varphi) = Const_{n,\ell}\, e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^{\ell} L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right) Y_\ell^m(\theta,\phi)$;

$L^2|\ell\, m_\ell\rangle = \ell(\ell+1)\hbar^2|\ell\, m_\ell\rangle$, $L_z|\ell\, m_\ell\rangle = m_\ell\hbar|\ell\, m_\ell\rangle$;

$S^2|s\, m_s\rangle = s(s+1)\hbar^2|s\, m_s\rangle$, $S_z|s\, m_s\rangle = m_s\hbar|s\, m_s\rangle$;

$J^2|j\, m_j\rangle = j(j+1)\hbar^2|j\, m_j\rangle$, $J_z|j\, m_j\rangle = m_j\hbar|j\, m_j\rangle$.

Code letters: "s" means $\ell = 0$, "p" means $\ell = 1$, "d" means $\ell = 2$, "f" means $\ell = 3$, etc.

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} \cong \frac{1}{137.036}$$

Spin-1/2

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Perturbation theory

$\mathrm{H}^0\psi_n^0 = E_n^0\psi_n^0$, $H = H^0 + H^1$, $\mathrm{H}\psi_n = E_n\psi_n$

$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \ldots$; $E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots$;

$$E_n^1 = \int \psi_n^{0*}\, \mathrm{H}'\, \psi_n^0\, d^3r$$

$$\psi_n^1 = \sum_{\ell \neq n} \left(\frac{\int \psi_\ell^{0*}\, \mathrm{H}'\, \psi_n^0\, d^3r}{E_n^0 - E_\ell^0} \right) \psi_\ell^0,$$

$$E_n^2 = \sum_{k \neq n} \frac{\left| \int \psi_k^{0*}\, \mathrm{H}'\, \psi_n^0\, d^3r \right|^2}{E_n^0 - E_k^0},$$

$\overline{\overline{W}}\vec{\alpha} = E^1\vec{\alpha} \qquad W_{k,j} \equiv \langle \psi_k^0 | H' | \psi_j^0 \rangle$

Spin-orbit, Zeeman, Clebsch-Gordan

$\mathrm{H}_{so} = -\vec{\mu} \bullet \vec{B}$; $\vec{S} \bullet \vec{L} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$;

$$\left| j\ m_j \right\rangle = \sum_{m_\ell + m_s = m_j} C^{\ell\ \ s\ \ j}_{m_\ell\ m_s\ m_j} \left| \ell\ m_\ell \right\rangle \left| s\ m_s \right\rangle; \quad \vec{\mu}_{Total} = \vec{\mu}_\ell + \vec{\mu}_S = -\frac{e}{2m}(\vec{L} + 2\vec{S}); \quad \mathcal{H}^1_Z = -\vec{\mu}_{Total} \cdot \vec{B}_{ext};$$

$$E^1_Z = \frac{e}{2m} \vec{B}_{ext} \cdot \langle \vec{J} + \vec{S} \rangle;$$

$$E^1_Z = \mu_B g_J B_{ext} m_J; \quad \mu_B = \frac{e\hbar}{2m} \quad g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)};$$

$$E^1_{Z,strong} = \frac{e}{2m} \vec{B}_{ext} \cdot \langle \vec{L} + 2\vec{S} \rangle = \mu_B B_{ext}(m_\ell + 2m_s); \quad \mathrm{H}_{HF} = -\vec{\mu}_e \bullet \vec{B}_{dip};$$

$$E^1_{n,0,0} = \frac{\mu_0 g e^2}{3 m_e m_p} \frac{\hbar^2}{\pi n^3 a^3} \begin{cases} 1/4 & \text{TRIPLET} \\ -3/4 & \text{SINGLET} \end{cases} \quad P\Psi(1,2) = \Psi(2,1) \quad P^2 = 1$$

Identical particles

$$\Psi^0_A(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right) \quad \Psi^0_S(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right)$$

$\langle (x_1 - x_2)^2 \rangle_\pm = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2$ with $\langle x \rangle_{ab} \equiv \int x\, \psi^*_a(x)\, \psi_b(x) dx$

Time-dependent perturbation theory

$\mathcal{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \quad \Psi(\vec{r}, t) = \psi(\vec{r}) e^{-iEt/\hbar} \quad \dot{c}_a = -\frac{i}{\hbar} \mathcal{H}'_{ab} e^{-i\omega_0 t} c_b, \quad \dot{c}_b = -\frac{i}{\hbar} \mathcal{H}'_{ba} e^{+i\omega_0 t} c_a, \mathcal{H}'_{ab} \equiv \langle \psi_a | \mathcal{H}' | \psi_b \rangle,$
$\omega_0 = (E_b - E_a)/\hbar$

Sinusoidal perturbation: $\mathcal{H}'(\vec{r}, t) = V(\vec{r}) \cos \omega t$. Two-level system: $c_a(0) = 1, c_b(0) = 0, c_b(t) = -\frac{i}{\hbar} \int_0^t \mathcal{H}'_{ba}(t')\, e^{i\omega_0 t'} dt'$ $\quad P_{a \to b}(t) = |c_b(t)|^2 \cong \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$ with $V_{ab} \equiv \langle \psi_a | V(\vec{r}) | \psi_b \rangle$;

Spontaneous emission rate: $A = \frac{\omega_0^3 |\vec{\wp}|^2}{3\pi\varepsilon_0 \hbar c^3}$ with $|\vec{\wp}| \equiv q \langle \psi_b | \vec{r} | \psi_a \rangle$

Electric dipole selection rules: No transitions occur unless $\Delta m = \pm 1$ *or* 0 and $\Delta \ell = \pm 1$

Quantum statistical mechanics

Distinguishable Particles: $n_s = g_s e^{-(E_s - \mu)/k_B T}$,

Indistinguishable Bosons: $n_s = \frac{g_s}{e^{(E_s - \mu)/k_B T} - 1}$,

Indistinguishable Fermions: $n_s = \frac{g_s}{e^{(E_s - \mu)/k_B T} + 1}$

Planck blackbody radiation: $\rho(\omega) = \frac{\hbar \omega^3/(\pi^2 c^3)}{e^{\hbar\omega/k_B T} - 1}$;

Photon density of states: $g(\omega) = \frac{V}{\pi^2} \frac{\omega^2}{c^3}$;

Phonon energy: $U = 9Nk_B T \left(\frac{T}{\Theta_D}\right)^3 \int_0^{\Theta_D/T} \frac{x^3 dx}{e^x - 1}$;

Debye temperature $\Theta_D = \frac{\hbar \omega_{max}}{k_B}$;

Low temperature ($T \ll \Theta_D$) heat capacity $C_V = \frac{\partial U}{\partial T}\Big|_V = \frac{12 N k_B \pi^4}{5} \left(\frac{T}{\Theta_D}\right)^3$;

Particles of mass m in a 3D infinite square well of sides $L \times L \times L$ have eigenstates:
$\vec{k} = \frac{\pi}{L}(n_x, n_y, n_z), n_x = 1, 2, \ldots etc.$ and energy $E = \hbar^2 k^2 / 2m$

Fermi function, or occupation index, $\frac{n(E)}{g(E)} = \frac{1}{e^{(E-\mu)/k_B T} + 1}$;

$$\frac{n(E)}{g(E)} = \begin{cases} 1 & \text{for } E < \mu \\ 0 & \text{for } E > \mu \end{cases} \quad (T = 0).$$

Fermi energy $E_F = \frac{h^2}{8m_e} \left(\frac{3N/V}{\pi}\right)^{2/3}$

WKB

$$\psi(x) = \frac{D}{\sqrt{p_{class}(x)}} \exp\left[\pm \frac{i}{\hbar}\int^{x} p_{class}(x')dx'\right]; \; p_{class} = \sqrt{2m(E - V(x))};$$

For infinite square well potentials: $\frac{1}{\hbar}\int_0^a \sqrt{2m(E_n - V(x))}\,dx = \pi n$, with $n = 1, 2, 3, ...$;

For finite wells: $\int_0^a \sqrt{2m(E_n - V(x))}\,dx = \pi\hbar\left(n - \frac{1}{2}\right)$, with $n = 1, 2, 3, ...$;

For tunneling: $\psi(x) = \frac{D}{\sqrt{|p_{class}(x)|}} \exp\left[\pm \frac{1}{\hbar}\int^{x} |p_{class}(x')|dx'\right]$; $T \propto e^{-2\gamma}$,

where $\gamma = \frac{1}{\hbar}\int_0^a |p_{class}(x')|dx'$;

Fowler-Nordheim tunneling: $T = exp\left[-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{\Phi^{3/2}}{e\mathcal{E}}\right]$

Aharonov-Bohm Effect : $\frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2 \Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}$; $\Psi(\vec{r},t) = e^{ig(\vec{r})}\Psi'(\vec{r},t)$, where

$g(\vec{r}) \equiv \frac{q}{\hbar}\int_{\text{Ref. Pt.}}^{\vec{r}} \vec{A}(\vec{r}) \bullet d\vec{l}$; $|\Psi|^2 = |\Psi_{FreeParticle}|^2 \cos^2\left(\frac{q\Phi_B}{2\hbar}\right)$; $\Phi_B = \oiint \vec{B} \bullet d\vec{A}$

Variational principle

$E_{ground\ state} \leq \langle\psi|H|\psi\rangle$ for any ψ

Mathematical Formulae

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$ $\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$;

$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$ $\quad \int_0^\infty x^n\, e^{-x/a}dx = n!\,a^{n+1}$

$\int_0^\infty x^{2n}\, e^{-x^2/a^2}dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$ $\quad \int_0^\infty x^{2n+1}\, e^{-x^2/a^2}dx = \frac{n!}{2}a^{2n+2}$

Integration by parts: $\int_a^b f\frac{dg}{dx}dx = -\int_a^b \frac{df}{dx}g\;dx + fg|_a^b$

$\int \sin\theta\cos\theta\;d\theta = -\int\cos\theta\;d(\cos\theta)$; $\int_0^\pi \sin^2(nu)\sin(u)du = \frac{4n^2}{4n^2 - 1}$; $\int_0^\infty r^n\,e^{-r/a}\,dr = a^{n+1}\,n!$

Binomial expansion for x <<1: $(1 + x)^n \cong 1 + nx + \frac{n(n-1)}{2}x^2$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad (x < 1)$

$e^{-x} = 1 - x + x^2/2! + \cdots \quad (x \ll 1)$

$\sin\theta = \theta - \theta^3/3! + \cdots \quad (\theta \ll 1)$

$\cos\theta = 1 - \theta^2/2! + \cdots \quad (\theta \ll 1)$