Physics 402
Spring 2019
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Formula Sheet Midterm 2

Infinite square well $\mathcal{H}^0 = \frac{p^2}{2m} + V(x)$ $\quad V(x) = \begin{cases} 0 \text{ for } 0 < x < a \\ \infty \text{ for x<0 and x>a} \end{cases}$ $\quad E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$

and $\psi^0(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \text{ for } 0 < x < a \\ 0 \text{ for x<0 and x>a} \end{cases}$ with $n = 1, 2, 3, \ldots$

Harmonic Oscillator $H^0 = \frac{p_x^2}{2m} + \frac{k}{2}x^2$, $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$, $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$,

$E_n = (n + \frac{1}{2})\hbar\omega;\ n = 0, 1, 2, \ldots;$

$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-);\ p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-);\ a_+\psi_n = \sqrt{n+1}\psi_{n+1};\ a_-\psi_n = \sqrt{n}\psi_{n-1}$

Hydrogen Atom $\mathrm{H}^0 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{(-e)(+2e)}{4\pi\varepsilon_0 r}$, $n = 1, 2, 3, \ldots, \ell = 0, 1, \ldots, n-1,$

$-\ell \le m \le \ell,\ \psi^0_{n,\ell,m}(r,\theta,\varphi) = Const_{n,\ell}\, e^{-\frac{r}{na}}\left(\frac{2r}{na}\right)^\ell L^{2\ell+1}_{n-\ell-1}\left(\frac{2r}{na}\right)Y_\ell^m(\theta,\phi);$

$L^2|\ell\, m_\ell\rangle = \ell(\ell+1)\hbar^2|\ell\, m_\ell\rangle,\ L_z|\ell\, m_\ell\rangle = m_\ell\hbar|\ell\, m_\ell\rangle;\ S^2|s\, m_s\rangle = s(s+1)\hbar^2|s\, m_s\rangle,$

$S_z|s\, m_s\rangle = m_s\hbar|s\, m_s\rangle;\ J^2|j\, m_j\rangle = j(j+1)\hbar^2|j\, m_j\rangle,\ J_z|j\, m_j\rangle = m_j\hbar|j\, m_j\rangle$

Spin-1/2 $\quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Perturbation theory $\mathrm{H}^0\psi_n^0 = E_n^0\psi_n^0,\ H = H^0 + H^1,\ \mathrm{H}\psi_n = E_n\psi_n$

$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \ldots;\ E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots;\ E_n^1 = \int \psi_n^{0*}\,\mathrm{H}'\,\psi_n^0\, d^3r$

$$\psi_n^1 = \sum_{\ell\ne n}\left(\frac{\int \psi_\ell^{0*}\,\mathrm{H}'\,\psi_n^0\, d^3r}{E_n^0 - E_\ell^0}\right)\psi_\ell^0,\ E_n^2 = \sum_{k\ne n}\frac{\left|\int \psi_k^{0*}\,\mathrm{H}'\,\psi_n^0\, d^3r\right|^2}{E_n^0 - E_k^0},\ \mathrm{H}' = -\frac{p^4}{8m^3c^2};$$

$$E^1_{n,\ell} = -|E_n^0|\frac{\alpha^2}{4n^2}\left[\frac{4n}{\ell+\frac{1}{2}} - 3\right] \quad \alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} \cong \frac{1}{137.036} \quad \bar{\bar{W}}\vec{\alpha} = E^1\vec{\alpha} \quad W_{k,j} \equiv \langle\psi_k^0|H'|\psi_j^0\rangle$$

$$\mathrm{H}_{so} = -\vec{\mu}\bullet\vec{B};\ \vec{S}\bullet\vec{L} = \frac{1}{2}\left(J^2 - L^2 - S^2\right);\ \Delta E = E_n^{1\,Relativity} + E_n^{1\,SpinOrbit} = \frac{|E_n^0|\alpha^2}{n^2}\left[\frac{3}{4} - \frac{n}{j+\frac{1}{2}}\right];$$

$$E_{fs}^1 = \frac{|E_n^0|\alpha^2}{n^3}\left\{\frac{3}{4n} - \left[\frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]\right\}\left|\, j\ m_j\right\rangle = \sum_{m_\ell+m_s=m_j} C^{\ell\ \ s\ \ j}_{m_\ell\ m_s\ m_j}\left|\ell\ m_\ell\right\rangle\left|s\ m_s\right\rangle;$$

$\vec{\mu}_{Total} = \vec{\mu}_\ell + \vec{\mu}_S = -\frac{e}{2m}\left(\vec{L} + 2\vec{S}\right)$

$\mathcal{H}_Z^1 = -\vec{\mu}_{Total}\cdot\vec{B}_{ext} \quad E_Z^1 = \frac{e}{2m}\vec{B}_{ext}\cdot\langle\vec{J} + \vec{S}\rangle \quad E_Z^1 = \mu_B g_J B_{ext} m_J$

$\mu_B = \frac{e\hbar}{2m}\ g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$

$$E^1_{Z,strong} = \frac{e}{2m}\vec{B}_{ext}\cdot\langle\vec{L}+2\vec{S}\rangle = \mu_B B_{ext}(m_\ell + 2m_s)\ ;$$

$$E^1_{n,0,0} = \frac{\mu_0 g e^2}{3m_e m_p}\frac{\hbar^2}{\pi n^3 a^3}\begin{cases}1/4 & \text{TRIPLET}\\ -3/4 & \text{SINGLET}\end{cases} \quad P\Psi(1,2) = \Psi(2,1) \quad P^2 = 1$$

Identical particles

$$\Psi^0_A(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right) \quad \Psi^0_S(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right)$$

$\langle(x_1 - x_2)^2\rangle_\pm = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x^2\rangle_a\langle x^2\rangle_b \mp 2|\langle x\rangle_{ab}|^2$ with $\langle x\rangle_{ab} \equiv \int x\,\psi^*_a(x)\,\psi_b(x)dx$

Some integrals

$$\int \sin\theta\cos\theta\,d\theta = -\int\cos\theta\,d(\cos\theta); \quad \int_0^\pi \sin^2(nu)\sin(u)du = \frac{4n^2}{4n^2-1}; \quad \int_0^\infty r^n e^{-r/a}\,dr = a^{n+1}\,n!$$

Time-dependent Perturbation theory

$\mathcal{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ $\Psi(\vec{r},t) = \psi(\vec{r})e^{-iEt/\hbar}$ $\dot{c}_a = -\frac{i}{\hbar}\mathcal{H}'_{ab}e^{-i\omega_0 t}c_b$, $\dot{c}_b = -\frac{i}{\hbar}\mathcal{H}'_{ba}e^{+i\omega_0 t}c_a$,

$\mathcal{H}'_{ab} \equiv \langle\psi_a|\mathcal{H}'|\psi_b\rangle$, $\omega_0 = (E_b - E_a)/\hbar$

Sinusoidal perturbation: $\mathcal{H}'(\vec{r},t) = V(\vec{r})\cos\omega t$.

Two-level system: $c_a(0) = 1$, $c_b(0) = 0$, $c_b(t) = -\frac{i}{\hbar}\int_0^t \mathcal{H}'_{ba}(t')\,e^{i\omega_0 t'}dt'$

$P_{a\to b}(t) = |c_b(t)|^2 \cong \frac{|V_{ab}|^2}{\hbar^2}\frac{sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$ with $V_{ab} \equiv \langle\psi_a|V(\vec{r})|\psi_b\rangle$

Fermi gas

Fermi radius: $k_F = (3\rho\pi^2)^{1/3}$; $E_F = \frac{\hbar^2 k_f^2}{2m}$; ρ = free electron density