Vector Spaces and Hilbert Spaces (Mainly, Griffiths A.1, A.2)

A **vector space** is a set of objects called vectors ($|\mathbf{A}\rangle$, $|\mathbf{B}\rangle$, $|\mathbf{C}\rangle$,...) and a set of numbers called scalars (a, b, c,...) along with a rule for vector addition and a rule for scalar multiplication. If the scalars are real, we have a **real vector space**; if the scalars are complex, we have a **complex vector space**. The set must be **closed** under vector addition and scalar multiplication.

Vector addition must have these properties:

- The sum of any 2 vectors is a vector: $|\mathbf{A}\rangle + |\mathbf{B}\rangle = |\mathbf{C}\rangle$
- Vector addition is commutative and associative:

$$|\mathbf{A}\rangle + |\mathbf{B}\rangle = |\mathbf{B}\rangle + |\mathbf{A}\rangle \quad \text{and} \quad |\mathbf{A}\rangle + (|\mathbf{B}\rangle + |\mathbf{C}\rangle) = (|\mathbf{A}\rangle + |\mathbf{B}\rangle) + \mathbf{C}\rangle$$

- There exists a zero vector $|\mathbf{0}\rangle$ such that : $|\mathbf{A}\rangle + |\mathbf{0}\rangle = |\mathbf{A}\rangle$ for any vector $|\mathbf{A}\rangle$
- For every vector $|\mathbf{A}\rangle$ there is an inverse vector $|\mathbf{-A}\rangle$ such that $|\mathbf{A}\rangle + |\mathbf{-A}\rangle = |\mathbf{0}\rangle$

Scalar multiplication must have these properties:

- The product of a scalar and a vector is another vector: $b\,|\mathbf{A}\rangle = |\mathbf{C}\rangle$
- It is distributive with respect to vector addition and scalar addition:

$$a\,(|\mathbf{A}\rangle + |\mathbf{B}\rangle) = a\,|\mathbf{B}\rangle + a\,|\mathbf{A}\rangle \quad \text{and} \quad (a + b)\,|\mathbf{A}\rangle = a\,|\mathbf{A}\rangle + b\,|\mathbf{A}\rangle$$

- It is associative with respect to ordinary scalar multiplication: $a\,(b\,|\mathbf{A}\rangle) = (a\,b)\,|\mathbf{A}\rangle$
- Multiplication by the scalars 0 and 1 yields the expected: $0\,|\mathbf{A}\rangle = |\mathbf{0}\rangle$ and $1\,|\mathbf{A}\rangle = |\mathbf{A}\rangle$

Any collection of objects which obeys these rules is a vector space. Notice that this definition of a vector space does **not** involve definitions of either vector direction or vector magnitude. We are used to thinking of vectors as arrow-like things with both direction and magnitude, but for a general vector space, this may not be so.

In addition to the rules above, if there is a rule defining an **inner product,** then we have a kind of vector space called an **inner product space**. The inner product of two vectors $|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ is a complex number, written $\langle\mathbf{A}|\mathbf{B}\rangle$ with the following properties:

$$\langle\mathbf{B}|\mathbf{A}\rangle = \langle\mathbf{A}|\mathbf{B}\rangle^* \;,\; \langle\mathbf{A}|\mathbf{A}\rangle \geq 0 \;,\; \langle\mathbf{A}|\mathbf{A}\rangle = 0 \Leftrightarrow |\mathbf{A}\rangle = |\mathbf{0}\rangle \;,\; \langle\mathbf{A}|\,(\,b\,|\mathbf{B}\rangle + c\,|\mathbf{C}\rangle\,) = b\,\langle\mathbf{A}|\mathbf{B}\rangle + c\,\langle\mathbf{A}|\mathbf{C}\rangle$$

Finally, the collection of all complex square-integrable functions f(x) , with inner product $\langle f\,|\,g\rangle = \int dx\, f(x)^* g(x)$, is an inner product space called a **Hilbert Space.** Square-integrable means $\int dx\, |f(x)|^2 < \infty$. Usually the functions are defined over all space and the integrals are over all space ($-\infty<x<\infty$). But sometimes, the functions are only defined over a finite range ($a < x < b$) and then the integrals are only over that range $\int_a^b dx\,(...)$. An example of a finite-range Hilbert Space is the wavefunctions for the infinite square well (which only exist inside the well $0 < x < a$).