1. Prove that for the harmonic oscillator $\hat{a}_+ \hat{a}_- = \frac{1}{\hbar \omega} \hat{p} - \frac{1}{2}$ starting with the definitions of the $\hat{a}_+$ and $\hat{a}_-$ operators in terms of the position and momentum operators.

2. Prove that $\hat{H}(\hat{a}_- \psi) = (E - \hbar \omega)(\hat{a}_- \psi)$, i.e. the “lowered” version of a harmonic oscillator solution $\psi(x)$ has an energy value that is $\hbar \omega$ lower than that of $\psi$.

3. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.10


5. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.13 \{Hint: in part (c) use Ehrenfest’s theorem Eq. (1.33) to get $\langle p \rangle$ from $\langle x \rangle$.\}

6. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.41 \{Hint: No calculations needed! Think about the effects of the new boundary condition imposed by this potential compared to the original harmonic oscillator.\}

7. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.58 \{Hint: Mathematica or WolframAlpha code: Sum[n^2, {n, 1, N}]\}

8. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.20

9. What follows is a way to “derive” quantum mechanics starting from classical mechanics, following the logic of P. A. M. Dirac. We consider first the Poisson Bracket (PB) from classical physics, which is defined as follows. Consider two dynamical functions of the generalized coordinates and conjugate momenta of a single particle: $g(\vec{q}, \vec{p})$ and $h(\vec{q}, \vec{p})$, where $\vec{q} = (q_1, q_2, q_3)$ and $\vec{p} = (p_1, p_2, p_3)$ is the position and momentum of a single particle in three dimensions. Examples of $g$ and $h$ include components of angular momentum, a component of linear momentum, mechanical energy, linear kinetic energy, rotational kinetic energy, etc. Define the PB of $g, h$ as $[g, h] \equiv \sum_{i=1}^{n} \left( \frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$, where $n$ is the dimensionality of the system. (Note that $q_1 = x, p_1 = p_x$, etc.)
   a) Prove from the definition of the PB that $[g, h] = -[h, g]$.
   b) Show that for a single particle in 3 dimensions $[q_j, q_k] = 0, [p_j, p_k] = 0$, and most
   Continued on the next page
Interestingly \([q_j, p_k] = \delta_{kj}\), where \(j, k \in \{1, 2, 3\}\) and \(n = 3\), and \(\delta_{kj}\) is the Kronecker delta. \(\{\text{Hint: use the fact that } \vec{q}, \vec{p}\ \text{are the independent variables that describe the state of the particle, i.e } \frac{\partial q_j}{\partial p_k} = 0, \text{ etc.}\}\)

If the PB of two dynamical quantities vanishes, then the quantities are said to commute. If the PB of two dynamical quantities is equal to 1, then the quantities are said to be canonically conjugate.

Starting with this, Dirac noted that the essential new ingredient of quantum mechanics (QM) is that certain observables [represented by operators \((\hat{u}, \hat{v})\)] give different answers depending on the order in which the observables operate on a QM wavefunction, or in other words \(\hat{u} \hat{v} \neq \hat{v} \hat{u}\). To account for this, Dirac re-defined the PB for the quantum case as follows: \(i\hbar [u, v] \equiv \hat{u} \hat{v} - \hat{v} \hat{u}\). This leads to the following statements of the “fundamental quantum conditions” for the quantum position and momentum operators: \(\hat{q}_r \hat{q}_s - \hat{q}_s \hat{q}_r = 0\), \(\hat{p}_r \hat{p}_s - \hat{p}_s \hat{p}_r = 0\), and \(\hat{q}_r \hat{p}_s - \hat{p}_s \hat{q}_r = i\hbar \delta_{rs}\). From this statement, one can derive many important results in quantum mechanics, as outlined in Dirac’s book Principles of Quantum Mechanics.

c) Given the definitions and Dirac’s argument above, find the quantum mechanical commutation relations corresponding to these pairs of classical dynamical functions: Two Cartesian components of the angular momentum vector in 3-dimensions, namely \(L_x\) and \(L_y\) (see Eq. (4.96)); \(L_z\) and \(x\); \(L_z\) and \(p_x\). Check your results against the quantum commutators given in Eqs. (4.99) and Eqs. (4.122).

**EXTRA CREDIT**

4. Griffiths and Schroeter Quantum Mechanics, 3rd Ed., Problem 2.49. This is a very interesting exact solution of the time-dependent Schrödinger equation for the harmonic oscillator.