Schrödinger's Trick

The time-dependent Schrödinger equation for the harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}Kx^2\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad \textbf{[1]}$$

whose stationary, bound-state solutions are

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

where $\psi(x)$ satisfies the time-independent equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}Kx^2\psi(x) = E\psi(x) \qquad \textbf{[2]}$$

It is not obvious how to solve Equation 2 for the allowed values of E and the corresponding wave functions $\psi(x)$. There are several general techniques for solving differential equations; however, this problem can be solved (exactly!) using a beautiful trick invented by Schrödinger.

Recalling that $\omega = \sqrt{K/m}$, we define $y = \sqrt{m\omega/\hbar}x$ and, correspondingly, $dy = \sqrt{m\omega/\hbar}\,dx$. Note that ω is the classical oscillator's angular frequency: $x = x_0 \cos \omega t$, which satisfies $m(d^2x/dt^2) = -Kx$. Therefore, substituting x and dx in terms of y and dy from above into Equation 2, we obtain

$$-\frac{\hbar^2}{2m}\frac{1}{(\sqrt{\hbar/m\omega})^2}\frac{d^2\psi}{dy^2} + \frac{1}{2}(m\omega^2)\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2 y^2\psi = E\psi$$

and

$$\frac{d^2\psi}{dy^2} - y^2\psi = -\frac{2E}{\hbar\omega}\psi \quad \text{or} \quad \left[\frac{d^2}{dy^2} - y^2\right]\psi = -\frac{2E}{\hbar\omega}\psi \qquad \textbf{[3]}$$

This can be written as

$$\left[\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right) - 1\right]\psi = -\frac{2E}{\hbar\omega}\psi \qquad \textbf{[4]}$$

To see that this is true, note that

$$\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi - \psi = \left(\frac{d}{dy} - y\right)\left(\frac{d\psi}{dy} + y\psi\right) - \psi$$

$$= \frac{d^2\psi}{dy^2} - y\frac{d\psi}{dy} + y\frac{d\psi}{dy} + y - y^2\psi - \psi = \frac{d^2\psi}{dy^2} - y^2\psi$$

So the Schrödinger equation for the harmonic oscillator becomes

$$\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi = \left(1 - \frac{2E}{\hbar\omega}\right)\psi \qquad \textbf{[5]}$$

Operating on Equation 5 from the left with $\left(\frac{d}{dy} + y\right)$, we obtain

$$\left(\frac{d}{dy} + y\right)\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi = \left(1 - \frac{2E}{\hbar\omega}\right)\left(\frac{d}{dy} + y\right)\psi$$

But, for any function f

$$\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)f = \left(\frac{d}{dy} - y\right)\left(\frac{df}{dy} + yf\right)$$

$$= \frac{d^2f}{dy^2} + y\frac{df}{dy} - y\frac{df}{dy} - f - y^2 f = \left(\frac{d^2}{dy^2} - y^2 - 1\right)f$$

This is true for any function $f(y)$, in particular for $f(y) = \left(\frac{d}{dy} + y\right)\psi$. Therefore,

$$\left(\frac{d^2}{dy^2} - y^2\right)\left(\frac{d}{dy} + y\right)\psi - \left(\frac{d}{dy} + y\right)\psi = \left(1 - \frac{2E}{\hbar\omega}\right)\left(\frac{d}{dy} + y\right)\psi$$

Rearranging this gives us

$$\left(\frac{d^2}{dy^2} - y^2\right)\left[\left(\frac{d}{dy} + y\right)\psi\right] = -\frac{2(E - \hbar\omega)}{\hbar\omega}\left[\left(\frac{d}{dy} + y\right)\psi\right] \qquad \textbf{[6]}$$

But recalling Equation 3, which is

$$\left[\frac{d^2}{dy^2} - y^2\right]\psi = -\frac{2E}{\hbar\omega}\psi$$

we see that, if we define $\psi' = \left(\frac{d}{dy} + y\right)\psi$ and $E' = E - \hbar\omega$, then Equation 6 becomes Equation 7:

$$\left[\frac{d^2}{dy^2} - y^2\right]\psi' = -\frac{2E'}{\hbar\omega}\psi' \qquad \textbf{[7]}$$

Thus, Equations 3 and 7 have the exact same form. This means that *if* we have found a solution $\psi(y)$ corresponding to energy E, *then* $((d/dy) + y)\psi = (d\psi/dy) + y\psi$ is *also* a solution, and its corresponding energy will be $(E - \hbar\omega)$. We can just keep going like this and each time the energy is lowered by $\hbar\omega$. This means that the spacing of the energy levels of the quantum harmonic oscillator is $\hbar\omega$.