# Review of Coordinate Systems

A good understanding of coordinate systems can be very helpful in solving problems related to Maxwell's Equations. The three most common coordinate systems are rectangular (*x, y, z*), cylindrical (*r, ϕ, z*), and spherical (r, θ, ϕ).

<u>Unit vectors in rectangular, cylindrical, and spherical coordinates</u>

In *rectangular coordinates* a point P is specified by *x, y,* and *z*, where these values are all measured from the origin (see figure at right). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ (also called $\hat{x}, \hat{y},$ and $\hat{z}$). The unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ form a right-handed set; that is, if you push $\hat{i}$ into $\hat{j}$ with your right hand, your right thumb will point along $\hat{k}$ direction.
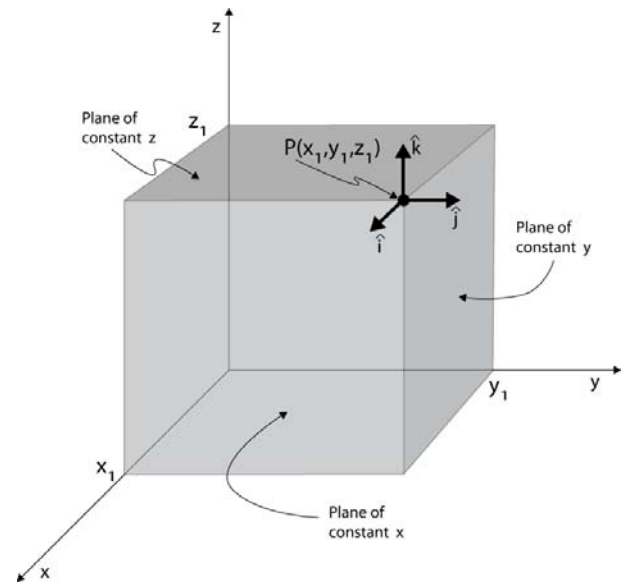
In *cylindrical coordinates* a point P is specified by *r, ϕ, z*, where ϕ is measured from the *x* axis (or *x-z* plane) (see figure at right). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{r}$ perpendicular to the cylinder of radius *r*, $\hat{\phi}$ perpendicular to the plane through the *z* axis at angle ϕ, and $\hat{z}$ perpendicular to the *x-y* plane at distance *z*. The unit vectors $\hat{r}, \hat{\phi}, \hat{z}$ form a right-handed set.
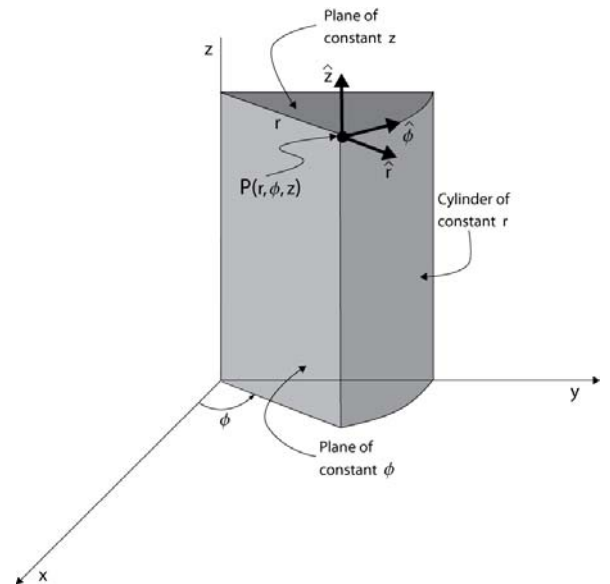
In *spherical coordinates* a point P is specified by r, θ, ϕ, where r is measured from the origin, θ is measured from the z axis, and ϕ is measured from the x axis (or x-z plane) (see figure at right). With z axis up, θ is sometimes called the *zenith* angle and ϕ the *azimuth* angle. A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{r}}$ perpendicular to the sphere of radius r, $\hat{\boldsymbol{\theta}}$ perpendicular to the cone of angle θ, and $\hat{\boldsymbol{\phi}}$ perpendicular to the plane through the z axis at angle ϕ. The unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ form a right-handed set.
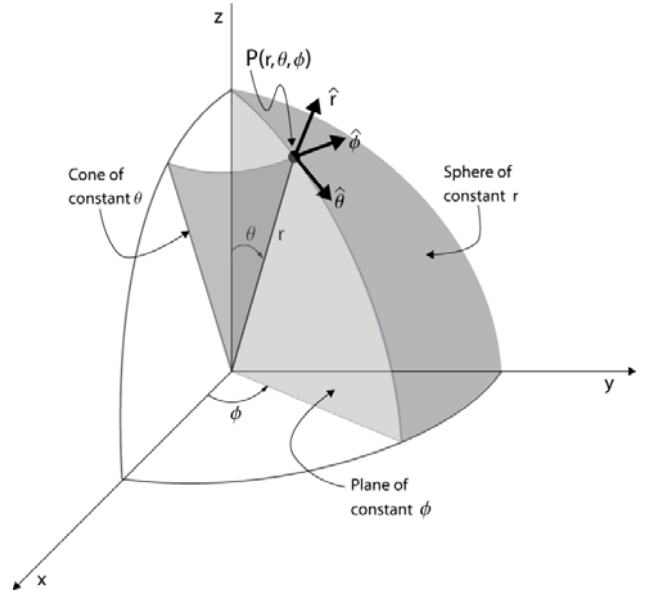
Infinitesimal lengths and volumes

An infinitesimal length in the *rectangular* system is given by

$$dL = \sqrt{dx^2 + dy^2 + dz^2} \tag{1}$$

and an infinitesimal volume by

$$dv = dx\,dy\,dz \tag{2}$$

In the *cylindrical* system the corresponding quantities are

$$dL = \sqrt{dr^2 + r^2\,d\phi^2 + dz^2} \tag{3}$$

and $$dv = dr\,r\,d\phi\,dz \tag{4}$$

In the *spherical* system we have

$$dL = \sqrt{dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2} \tag{5}$$

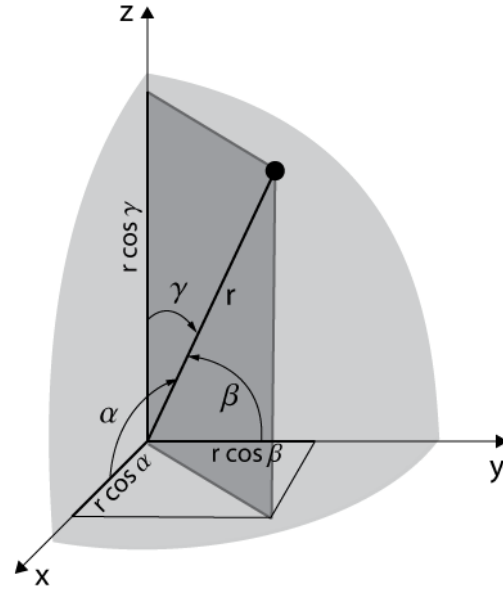and $$dv = dr\,r\,d\theta\,r\sin\theta\,d\phi \tag{6}$$

Direction cosines and coordinate-system transformation

As shown in the figure on the right, the
projection x of the scalar distance r on the x axis
is given by $r \cos \alpha$ where α is the angle
between r and the x axis. The projection of r on
the y axis is given by $r \cos \beta$, and the
projection on the z axis by $r \cos \gamma$. Note that
$\gamma = \theta$ so $\cos \gamma = \cos \theta$.

The quantities $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are
called the *direction cosines*. From the theorem
of Pythagoras,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad (7)$$

The scalar distance r of a spherical coordinate
system transforms into rectangular coordinate
distance

$$x = r \cos \alpha = r \sin \theta \cos \phi \qquad (8)$$
$$y = r \cos \beta = r \sin \theta \sin \phi \qquad (9)$$
$$z = r \cos \gamma = r \cos \theta \qquad (10)$$

from which

$$\cos \alpha = \sin \theta \cos \phi \qquad (11)$$
$$\cos \beta = \sin \theta \sin \phi \quad \text{direction cosines} \qquad (12)$$
$$\cos \gamma = \cos \theta \qquad (13)$$

As the converse of (8), (9), and (10), the spherical coordinate values (r, θ, ϕ) may be
expressed in terms of rectangular coordinate distances as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad r \geq 0 \qquad (14)$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0 \leq \theta \leq \pi) \qquad (15)$$

$$\phi = \tan^{-1} \frac{y}{x} \qquad (16)$$

From these and similar coordinate transformations of spherical to rectangular and
rectangular to spherical coordinates, we may express a vector **A** at some point P with
spherical components A_r, A_θ, A_ϕ as the rectangular components A_x, A_y, and A_z, where

$$A_x = A_r \sin\theta \, \cos\phi + A_\theta \cos\theta \, \cos\phi - A_\phi \sin\phi \qquad (17)$$
$$A_y = A_r \sin\theta \, \sin\phi + A_\theta \cos\theta \, \sin\phi + A_\phi \cos\phi \qquad (18)$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta \qquad (19)$$

Note that the direction cosines are simply the dot products of the spherical unit vector $\hat{\mathbf{r}}$ with the rectangular unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$:

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = \sin\theta\cos\phi = \cos\alpha \qquad (20)$$
$$\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\theta\sin\phi = \cos\beta \qquad (21)$$
$$\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = \cos\theta = \cos\gamma \qquad (22)$$

These and other dot product combinations are listed in the following table:

		Rectangular			Cylindrical			Spherical		
	Σ	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\phi}}$
Rectangular	$\hat{\mathbf{x}}$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
	$\hat{\mathbf{y}}$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\hat{\mathbf{r}}$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\hat{\boldsymbol{\phi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\hat{\mathbf{r}}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\hat{\boldsymbol{\theta}}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\hat{\boldsymbol{\phi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

Note that the unit vectors $\hat{\mathbf{r}}$ in the cylindrical and spherical systems are *not* the same. For example,

Spherical	*Cylindrical*
$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = \sin\theta\cos\phi$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = \cos\phi$
$\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\theta\sin\phi$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi$
$\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = \cos\theta$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = 0$

In addition to rectangular, cylindrical, and spherical coordinate systems, there are many other systems such as the elliptical, spheroidal (both prolate and oblate), and paraboloidal systems. Although the number of possible systems is infinite, all of them can be treated in terms of a *generalized curvilinear coordinate system*.

The fundamental parameters of the rectangular, cylindrical, and spherical coordinate systems are summarized in the following table:

Coordinate system	Coordinates	Range	Unit vectors	Length elements	Coordinate surfaces	
Rectangular	x	$-\infty$ to $+\infty$	$\hat{\mathbf{x}}$ or $\hat{\mathbf{i}}$	dx	Plane	x=constant
	y	$-\infty$ to $+\infty$	$\hat{\mathbf{y}}$ or $\hat{\mathbf{j}}$	dy	Plane	y=constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$ or $\hat{\mathbf{k}}$	dz	Plane	z=constant
Cylindrical	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Cylinder	r=constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\phi}}$	$r\,d\phi$	Plane	ϕ=constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$	dz	Plane	z=constant
Spherical	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Sphere	r=constant
	θ	0 to π	$\hat{\boldsymbol{\theta}}$	$r\,d\theta$	Cone	θ=constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\phi}}$	$r\sin\theta\,d\phi$	Plane	ϕ=constant

The following two tables give the unit vector dot products in rectangular coordinates for both rectangular-cylindrical and rectangular-spherical coordinates.

Σ	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	$\dfrac{y}{\sqrt{x^2+y^2}}$	0
$\hat{\boldsymbol{\phi}}$	$\dfrac{-y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0
$\hat{\mathbf{z}}$	0	0	1

Rectangular-cylindrical product in rectangular coordinates

Example: $\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{y}} = \cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$

·	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{y}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{z}{\sqrt{x^2+y^2+z^2}}$
$\hat{\boldsymbol{\theta}}$	$\dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$\dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$-\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$
$\hat{\boldsymbol{\varphi}}$	$-\dfrac{y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0

Rectangular-spherical product in rectangular coordinates

$$\text{Example: } \hat{\mathbf{x}}\cdot\hat{\mathbf{r}} = \sin\theta\,\cos\phi = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

Here are the transformations of vector components between coordinate systems:

Rectangular to cylindrical

$$A_r = A_x \frac{x}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

$$A_z = A_z$$

Cylindrical to rectangular

$$A_x = A_r \cos\phi - A_\phi \sin\phi$$

$$A_y = A_r \sin\phi - A_\phi \cos\phi$$

$$A_z = A_z$$

Rectangular to spherical

$$A_r = A_x \frac{x}{\sqrt{x^2+y^2+z^2}} + A_y \frac{y}{\sqrt{x^2+y^2+z^2}} + A_z \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$A_\theta = A_x \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + A_y \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - A_z \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2+y^2}} + A_y \frac{x}{\sqrt{x^2+y^2}}$$

Spherical to rectangular

$$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$$
$$A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta$$

And here are expressions for the gradient, divergence, and curl in all three coordinate systems:

Rectangular coordinates

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical coordinates

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}rA_r + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}rA_\phi - \frac{\partial A_r}{\partial \phi}\right) = \begin{vmatrix} \hat{\mathbf{r}}\frac{1}{r} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\frac{1}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

Spherical coordinates

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}rA_\phi\right) + \hat{\boldsymbol{\varphi}}\frac{1}{r}\left(\frac{\partial}{\partial r}rA_\theta - \frac{\partial A_r}{\partial \theta}\right)$$