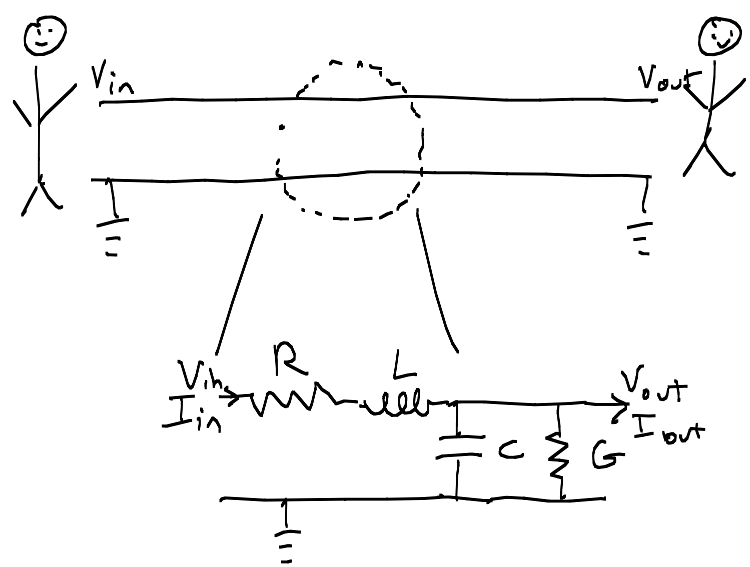
Telegrapher's Equation



$$\frac{\Delta V}{\Delta x} \longrightarrow \frac{\partial V}{\partial x} = -IR - L\frac{\partial I}{\partial t} \quad (1)$$

$$\frac{\Delta I}{\Delta x} \longrightarrow \frac{\partial I}{\partial x} = -GV - C\frac{\partial V}{\partial t} \quad (2)$$

$$\frac{\partial}{\partial x}(1): \quad \frac{\partial^2 V}{\partial x^2} = -\frac{\partial I}{\partial x}R - L\frac{\partial^2 I}{\partial x \partial t}$$

$$\frac{\partial}{\partial t}(2): \quad \frac{\partial^2 I}{\partial x \partial t} = -G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}$$

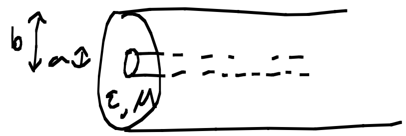
$$\frac{\partial^2 V}{\partial x^2} = -\left(-GV - C\frac{\partial V}{\partial t}\right)R - L\left(-G\frac{\partial V}{\partial t} - C\frac{\partial^2 V}{\partial t^2}\right)$$

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + CR)\frac{\partial V}{\partial t} + GRV$$

"Telegrapher's equation"

Simplest case: $R = G = 0$

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$ wave eqn $\frac{1}{s^2} = LC$ solutions: $V(x,t) = V(x \pm st)$ "d'Alembert"

what is s?: 

coaxial cable

Capacitance: $C = \frac{q}{V}$

Gauss' Law: $\oint \vec{E} \cdot d\vec{s} = \frac{q}{\varepsilon} \rightarrow E\, 2\pi r = \frac{q}{\varepsilon}$

$V = -\int \vec{E} \cdot d\vec{r} = -\int_a^b \frac{q}{2\pi r \varepsilon} dr = \frac{q}{2\pi\varepsilon} \ln \frac{b}{a}$

$C = \frac{q}{V} = \frac{2\pi\varepsilon}{\ln b/a}$

inductance: $L = \frac{\Phi}{I}$

Ampere's Law: $\oint \vec{B} \cdot d\vec{l} = \mu I$

$\rightarrow B\, 2\pi r = \mu I$

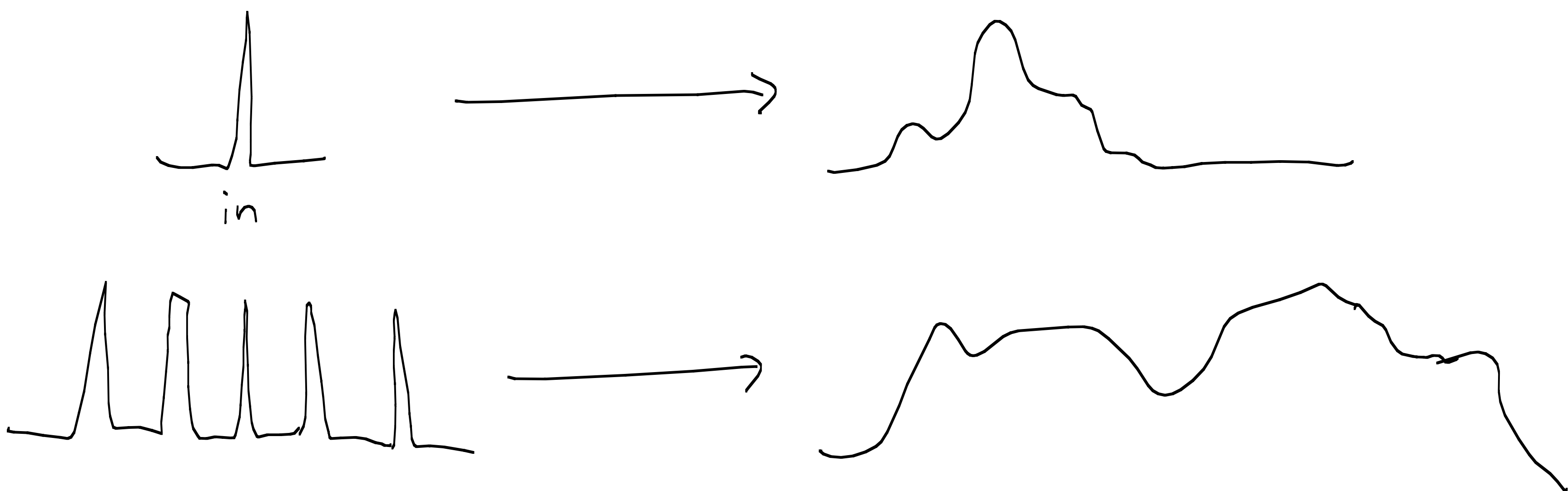
$\Phi = \int \vec{B} \cdot d\vec{s} = \int_a^b \frac{\mu I}{2\pi r} dr = \frac{\mu I}{2\pi} \ln \frac{b}{a}$

$L = \frac{\Phi}{I} = \frac{\mu}{2\pi} \ln \frac{b}{a}$

$$LC = \cancel{\frac{2\pi\varepsilon}{\ln b/a}} \frac{\mu}{\cancel{2\pi}} \cancel{\ln \frac{b}{a}} = \varepsilon\mu = \frac{1}{c^2/n^2} \rightarrow s = c/n$$

In general, $R, G \neq 0$

nonzero terms give attenuation and distortion



Can we choose L, C, G, R to eliminate distortion?

Ansatz

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG+CR)\frac{\partial V}{\partial t} + GRV$$ "Telegrapher's equation"

$$V(x,t) = u(x,t)\,e^{-\left(\frac{LG+CR}{2LC}\right)t}$$

$$\frac{\partial^2 u}{\partial x^2}e^{-(\ )t} = LC\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}e^{-(\ )t} - \frac{LG+CR}{2LC}u\,e^{-(\ )t}\right)$$
$$+ (LG+CR)\left(\frac{\partial u}{\partial t}e^{-(\ )t} - \frac{LG+CR}{2LC}u\,e^{-(\ )t}\right) + GRu\,e^{-(\ )t}$$

$$\frac{\partial^2 u}{\partial x^2}\cancel{e^{-(\ )t}} = LC\left[\frac{\partial^2 u}{\partial t^2}\cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}\frac{\partial u}{\partial t}\cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}\left(\frac{\partial u}{\partial t}\cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}\cancel{u\,e^{-(\ )t}}\right)\right]$$
$$+ (LG+CR)\left(\frac{\partial u}{\partial t}\cancel{e^{-(\ )t}} - \frac{LG+CR}{2LC}u\,\cancel{e^{-(\ )t}}\right) + GRu\,\cancel{e^{-(\ )t}}$$

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + \left(GR - \frac{(LG+CR)^2}{4LC}\right)u$$

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + \left(\frac{4GRLC - (L^2G^2 + 2LGCR + C^2R^2)}{4LC}\right)u$$

$$\underbrace{\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2}}_{\text{wave eqn.}} - \left(\frac{(LG-CR)^2}{4LC}\right)u$$

$$V(x,t) = u(x,t)\,e^{-\left(\frac{LG+CR}{2LC}\right)t}$$

If $LG = CR$, $u(x,t)$ is soln to wave equation

→ no distortion! (Heaviside 1893)

Submarine cables must be inductively loaded to increase L to satisfy Heaviside's requirement.