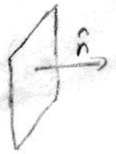
64a



$\vec{V}$ can be decomposed into a component perpendicular to the surface (parallel to $\hat{n}$), and a component parallel to surface (perpendicular to $\hat{n}$)

This corresponds to a component of flow across surface and a component along surface

$$\vec{V} = (\vec{V}\cdot\hat{n})\,\hat{n} - (\vec{V}\cdot\hat{n})\,\hat{n} + \vec{V}$$

65b-65c

$$\boxed{\Phi_E = \frac{q}{4\pi\varepsilon_0 R^2}\left(4\pi R^2\right) = \frac{q}{\varepsilon_0}}$$

66d-66e

$$\boxed{\vec{B} = m\,\frac{3\cos\theta\,\hat{r} - \hat{z}}{r^3}}$$

$$\Phi_B = m\int \frac{3\cos\theta(\hat{r}\cdot\hat{r}) - \hat{z}\cdot\hat{r}}{R^3}\,R^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{m}{R}\iint_0^{\pi} 2\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= 0$$

67a

$$\boxed{\frac{\partial r}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x^2+y^2}}\,2x = \frac{x}{r}}$$

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$\nabla\cdot\vec{V} = \nabla\cdot\left[f(r)\,\vec{r}\right]$$

$$= \frac{\partial}{\partial x}\left[x f(r)\right] + \frac{\partial}{\partial y}\left[y f(r)\right]$$

$$= f(r) + x\frac{\partial f}{\partial x} + f(r) + y\frac{\partial f}{\partial y}$$

$$= 2f(r) + \vec{r}\cdot\nabla f$$

; or using chain rule,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + y\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} = \frac{x^2}{r}\frac{\partial f}{\partial r} + \frac{y^2}{r}\frac{\partial f}{\partial r} = r\frac{\partial f}{\partial r}$$

$$= 2f(r) + r\frac{df}{dr}$$

68b

$$\boxed{\nabla \cdot \vec{v} = 0}$$

$$2f(r) + r\frac{df}{dr} = 0$$

$$r\frac{df}{dr} = -2f(r)$$

$$\frac{1}{f}df = -\frac{2}{r}dr$$

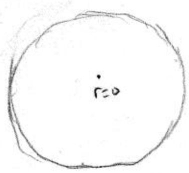
$$\ln f = -2\ln r + C$$

$$f = \frac{A}{r^2}$$

$$\vec{v} = f(r)\vec{r}$$

$$= \frac{A\vec{r}}{r^2}$$

68c



$\dot{V}$ is the volume of liquid flowing into circle at $r=0$

$$\dot{V} = \int \vec{v} \cdot d\vec{s}$$

$$= \int \frac{A}{r}\hat{r} \cdot d\vec{s}$$

$$= \frac{A}{r}(2\pi r)$$

$$A = \frac{\dot{V}}{2\pi}$$

$$\vec{v} = \frac{A\vec{r}}{r^2}$$

$$= \frac{\dot{V}\hat{r}}{2\pi r}$$

69a

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r} = \frac{q}{4\pi\varepsilon_0 r^3}\left[x\hat{x}+y\hat{y}+z\hat{z}\right] = \frac{q}{4\pi\varepsilon_0 r^3}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

69b

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x} = \frac{1}{r^3} - \frac{3x}{r^4}\frac{x}{r} = \frac{1}{r^5}\left[r^2-3x^2\right]$$

70c

$$\nabla\cdot\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^5}\left[r^2-3x^2+r^2-3y^2+r^2-3z^2\right] = \frac{q}{4\pi\varepsilon_0 r^5}\left[3r^2-3r^2\right]=0$$

70d

$$\vec{B} = \frac{m\,3\hat{r}(\hat{r}\cdot\hat{z})}{r^3} - \frac{m\hat{z}}{r^3} = \frac{3m\vec{r}(\vec{r}\cdot\hat{z})}{r^5} - \frac{m\hat{z}}{r^3}$$

71e

$$\vec{B} = \frac{3mz}{r^5}\begin{pmatrix}x\\y\\z\end{pmatrix} - \frac{m}{r^3}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{3mxz}{r^5}\\[4pt] \frac{3myz}{r^5}\\[4pt] \frac{3mz^2}{r^5}-\frac{m}{r^3}\end{pmatrix}$$

71f

$$\frac{\partial B_x}{\partial x} = 3m\left[\frac{z}{r^5} - \frac{5xz}{r^6}\frac{x}{r}\right] = 3m\left[\frac{z}{r^5} - \frac{5x^2z}{r^7}\right]$$

$$\frac{\partial B_y}{\partial y} = 3m\left[\frac{z}{r^5} - \frac{5y^2z}{r^7}\right]$$

$$\frac{\partial B_z}{\partial z} = 3m\left[\frac{2z}{r^5} - \frac{5z^3}{r^7}\right] + \frac{3m}{r^4}\frac{z}{r} = 3m\left[\frac{3z}{r^5} - \frac{5z^3}{r^7}\right]$$

71g

$$\nabla\cdot\vec{B} = 3m\left[\frac{5z}{r^5} - \frac{5zr^2}{r^7}\right]=0$$

72a

$$V_r(r+dr,\phi,z)\,[r+dr]\,d\phi\,dz - V_r(r,\phi,z)\,r\,d\phi\,dz$$
$$= \left[V_r(r,\phi,z) + \frac{\partial V_r}{\partial r}dr\right][r+dr]\,d\phi\,dz - V_r(r,\phi,z)\,r\,d\phi\,dz$$
$$= V_r\,dr\,d\phi\,dz + \frac{\partial V_r}{\partial r}\,r\,d\phi\,dz$$
$$= \frac{\partial}{\partial r}(rV_r)\,dr\,d\phi\,dz$$

72b

$$V_\phi(r,\phi+d\phi,z)\,dr\,dz - V_\phi(r,\phi,z)\,dr\,dz = \frac{\partial V_\phi}{\partial \phi}\,dr\,d\phi\,dz$$

72c

$$V_z(r,\phi,z+dz)\,dr\,r\,d\phi - V_z(r,\phi,z)\,r\,dr\,d\phi = \frac{\partial V_z}{\partial z}\,r\,dr\,d\phi$$

73d

$$dV = r\,dr\,d\phi\,dz$$

73e

$$\nabla\cdot\vec{V} = \frac{\text{Flux}}{\text{volume}} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_r) + \frac{\partial V_\phi}{\partial \phi} + r\frac{\partial V_z}{\partial z}\right] = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

73f

$$\vec{V}(\vec{r}) = f(r)\,\vec{r}$$

$$\nabla\cdot\vec{V} = \nabla\cdot[f(r)\,r\,\hat{r}] = \frac{1}{r}\frac{\partial}{\partial r}\left[f(r)\,r^2\right] = \frac{1}{r}\left[r^2\frac{\partial f}{\partial r} + 2r\,f(r)\right] = r\frac{df}{dr} + 2f(r)$$

73g

$$\nabla\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(V_r\,r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(V_\theta\,r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(V_\phi\,r\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial V_r}\left(V_r r^2\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(V_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$